In The Name of God The Most Compassionate, The Most Merciful

دانشگاه صنعتی شیراز

Linear Control Systems

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Root Locus Analysis

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Introduction

- The basic characteristic of the transient response of a closed-loop system is closely related to **the location of the closed-loop poles**.
- If the system has a **variable loop gain**, then the location of the closed-loop poles depends on the value of the loop gain chosen.
- It is important, therefore, that the designer know how the **closed-loop poles move** in the s plane as the loop gain is varied.
- Unless otherwise stated, we shall assume that the **gain of the open-loop transfer function** is the parameter to be varied through all values, from zero to infinity.

Definitions

- Generally in any transfer function, the **number of poles** (finite and infinite) is **equal** to the **number of zeros** (finite and infinite).

- **Example 1**: Find the finite and infinite poles and zeros of the following system

$$\frac{C(s)}{R(s)} = \frac{5(s+1)}{(s+3)(s+4)}$$

Finite zero is $z_1 = 1$

Infinite zero is $z_2 \to \infty$

Finite poles are $p_1 = 3$

$p_2 = 4$

Definitions

- **Example 2**: Find the finite and infinite poles and zeros of the following system

$$\frac{C(s)}{R(s)} = \frac{3(s+2)(s+6)}{(s-1)}$$

Finite zeros are $z_1 = 2$

$z_2 = 6$

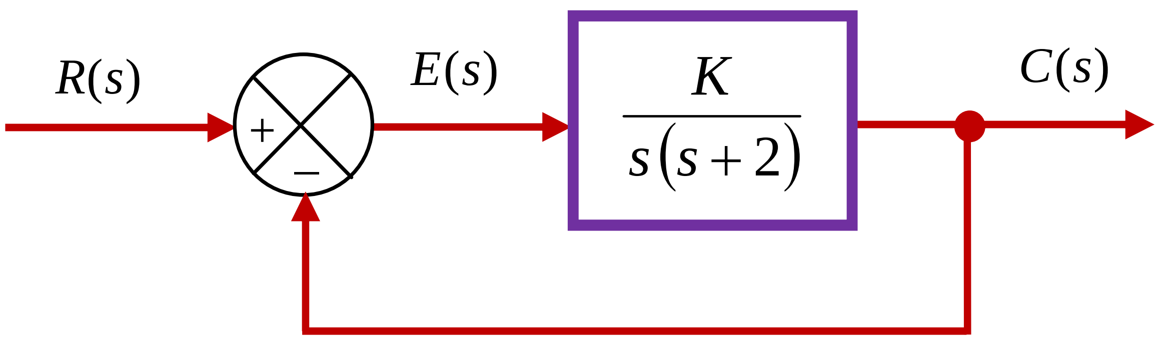
Finite pole is $p_1 = -1$

Infinite pole is $p_2 \to \infty$

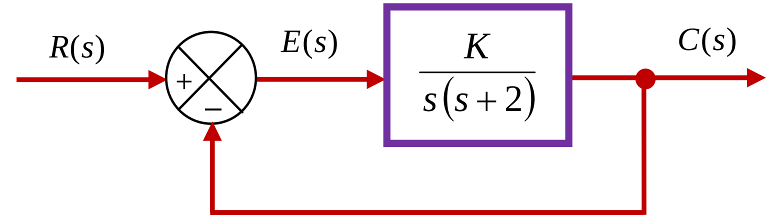
Root-Locus Plots

The aim is to find the location of closed-loop poles in *s* plane with respect to the variation of a parameter K.

Example: In the following system, find the location of closed-loop poles in s plane with respect to the negative or positive variation of a parameter K.



Root-Locus Plots



Solution:

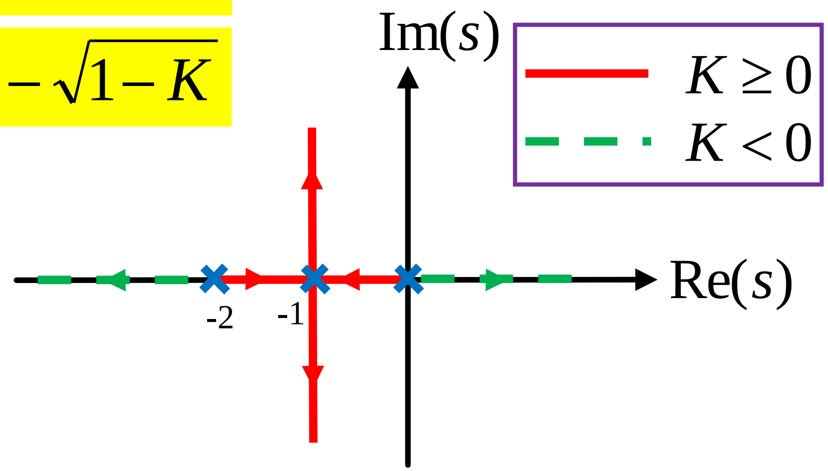
$$\frac{C(s)}{R(s)} = \frac{K}{s^2 + 2s + K}$$

$$\Delta = s^2 + 2s + K = 0$$

$$s_1 = -1 + \sqrt{1-K}$$

$$s_2 = -1 - \sqrt{1-K}$$

K	s_1	s_2
0	0	-2
0.5	-0.3	-1.7
1	-1	-1
2	-1+j	-1-j
5	-1+2j	-1-2j
-1	0.4	-2.4



$K > 0$ **stable** **Root Locus (RL)**

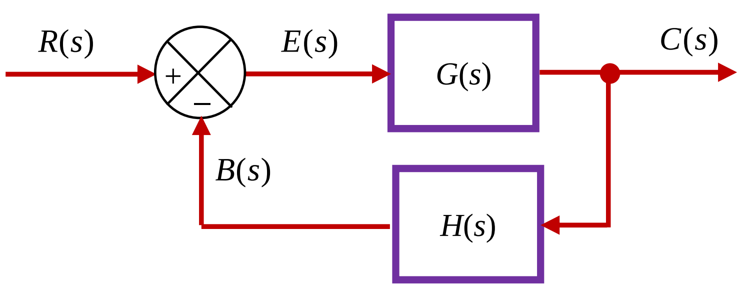
$K < 0$ **unstable** **Complementary Root Locus (CRL)**

Root-Locus Plots

The aim is to find the location of closed-loop poles in s plane with respect to the variation of a parameter K.

Angle and Magnitude Conditions. Consider the negative feedback system shown below. The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$



Root-Locus Plots

- The characteristic equation for this closed-loop system is obtained by setting the denominator equal to zero. That is,

$$1 + G(s)H(s) = 0$$

or

$$G(s)H(s) = -1 \qquad (1)$$

- Here we assume that $G(s)H(s)$ is a ratio of polynomials in s.
- Since $G(s)H(s)$ is a complex quantity, Equation (1) can be split into two equations by equating the angles and magnitudes of both sides, respectively, to obtain the following:

Root-Locus Plots

$$G(s)H(s) = -1$$

- **Angle condition**:

$$\angle G(s)H(s) = \pm 180^\circ (2l+1) \qquad l = 0,1,2,\ldots$$

- **Magnitude condition**:

$$|G(s)H(s)| = 1$$

The values of s that fulfill both the angle and magnitude conditions are the roots of the characteristic equation, or the closed-loop poles.

Root-Locus Plots

- In many cases, $G(s)H(s)$ involves a **gain parameter** K, and the characteristic equation may be written as

$$1+G(s)H(s)=0 \quad \Rightarrow \quad 1+K\frac{N(s)}{D(s)}=0 \quad \Rightarrow \quad 1+\frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}=0$$

- Then the root loci for the system are the **loci of the closed-loop poles** as the **gain K is varied** from zero to infinity.
- Note that to begin sketching the root loci of a system by the root-locus method we must know **the location of the poles and zeros** of $G(s)H(s)$.
- Remember that the angles of the complex quantities originating from the open-loop poles and open-loop zeros to the test point s are measured in the **counterclockwise** direction.

Root-Locus Plots

$$G(s)H(s) = -1 \quad \Rightarrow \quad K\frac{N(s)}{D(s)} = -1$$

- **Angle condition**:

$$\angle G(s)H(s) = \pm 180^{\circ}(2l+1) \qquad l = 0,1,2,\ldots$$

$$\Rightarrow \quad \angle N(s) - \angle D(s) = \pm 180^{\circ}(2l+1) \qquad l = 0,1,2,\ldots$$

- **Magnitude condition**:

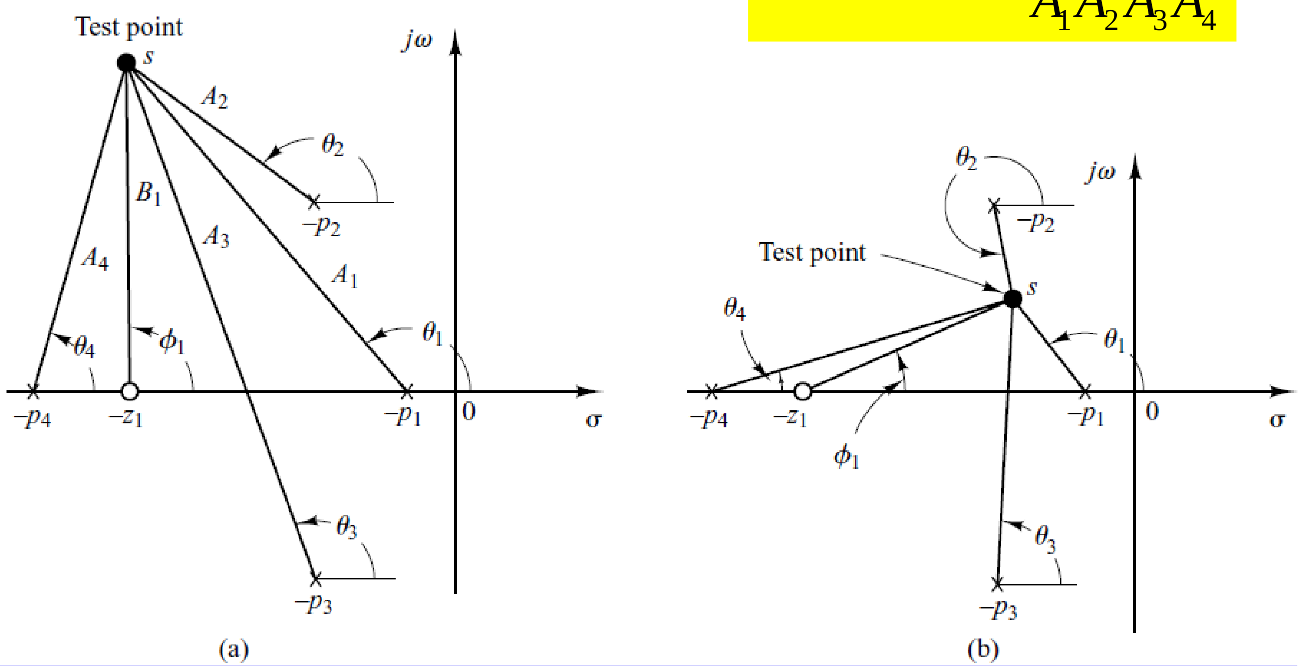
$$\left|G(s)H(s)\right| = 1 \quad \Rightarrow \quad \left|K\right| = \left|\frac{D(s)}{N(s)}\right|$$

Root-Locus Plots

Example, if $G(s)H(s)$ is given by

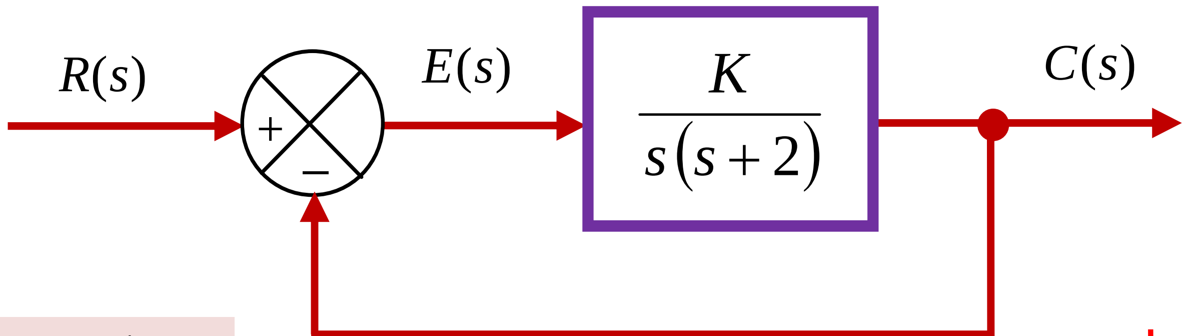
$$G(s)H(s)=\frac{K(s+z_1)}{(s+p_1)(s+p_2)(s+p_3)(s+p_4)}$$

- where $-p_2$ and $-p_3$ are complex-conjugate poles, then the angle of $G(s)H(s)$ is $$\angle G(s)H(s)=\phi_1-\theta_1-\theta_2-\theta_3-\theta_4$$
- The magnitude of $G(s)H(s)$ for this system is $$|G(s)H(s)|=\frac{KB_1}{A_1A_2A_3A_4}$$

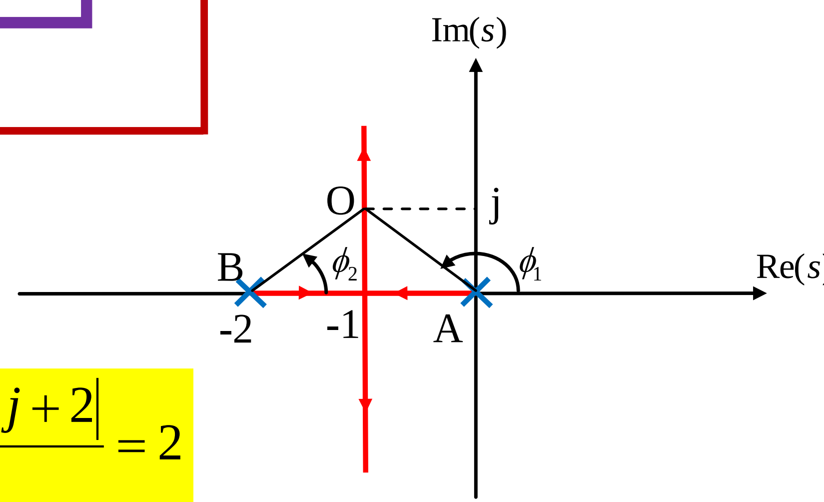


Root-Locus Plots

Example: In the following system, check the angle and magnitude conditions for *s* = -1+*j*.



$$\frac{N(s)}{D(s)} = \frac{1}{s(s+2)}$$



$$\left|K\right|_{s=-1+j} = \left|\frac{D(s)}{N(s)}\right|_{s=-1+j} = \frac{|OA||OB|}{1} = \frac{|-1+j||-1+j+2|}{1} = 2$$

$$\angle N(s) - \angle D(s) = \pm 180$$

$$0 - (\phi_1 + \phi_2) = \pm 180$$

$$0 - (135 + 45) = -180$$

Root-Locus Plots

Angle Condition

For all points on the root locus the angle condition is always satisfied.

Root-Locus Plots

Symmetry about the real axis

- Note that, because the open-loop complex-conjugate poles and complex-conjugate zeros, if any, are always located **symmetrically** about the real axis, the root loci are always symmetrical with respect to this axis.

- Therefore, we only need to construct the **upper half of the root loci** and draw the mirror image of the upper half in the lower-half s plane.

Procedure of Root-Locus Plots

1. Determine $\Delta(s)$ in the form of $1+K\dfrac{N(s)}{D(s)}$

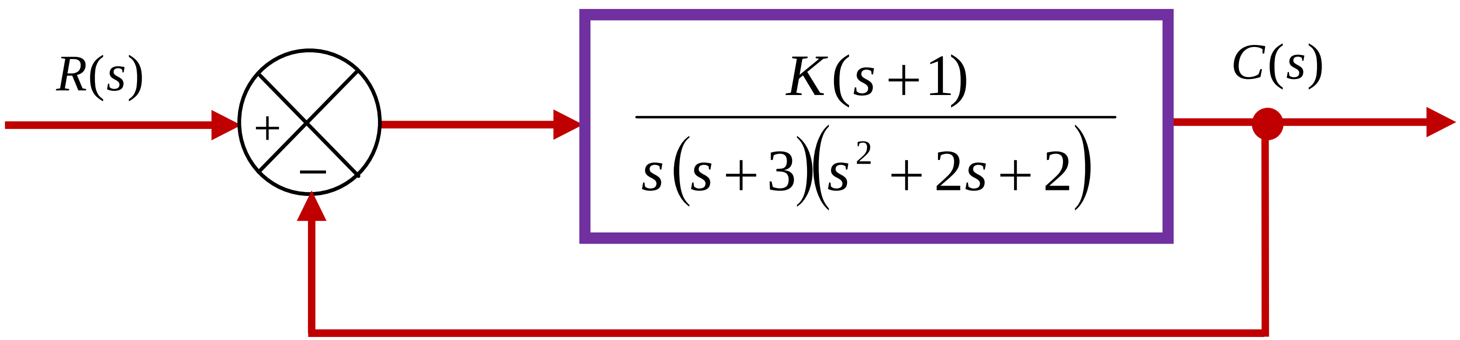
$$1+\frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}=0$$

2. Determine the finite poles and zeros of $\dfrac{N(s)}{D(s)}$

3. The root-locus **branches** **start from open-loop poles** and **terminate at zeros** (finite zeros or zeros at infinity). The number of branches is $\mathbf{max}(n,m)$ where n is the number of finite poles and m is the number of finite zeros.

Root-Locus Plots

Example: In the following system, find the location of closed-lop poles in *s* plane with respect to the positive variation of the parameter K.



Root-Locus Plots

Solution:

1. Determine $\Delta(s)$ in the form of $1+K\dfrac{N(s)}{D(s)}$

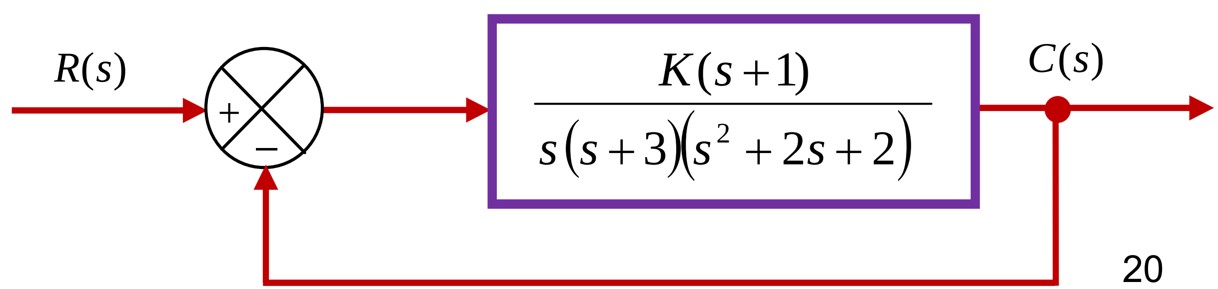
$$\Delta(s)=1+K\frac{s+1}{s(s+3)(s^2+2s+2)} \qquad \frac{N(s)}{D(s)}=\frac{s+1}{s(s+3)(s^2+2s+2)}$$

2. Determine the finite poles and zeros of $\dfrac{N(s)}{D(s)}$

Poles are $s_1=0$ $\quad s_2=-3$ $\quad s_{3,4}=-1\pm j$ $\quad n=4$

Zero is $s_1=-1$ $\quad m=1$

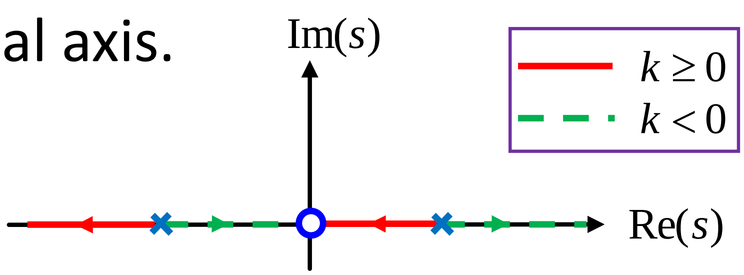
3. The number of branches is $\max(n,m) = 4$



Procedure of Root-Locus Plots

4. Determine the **root loci on the real axis**. Root loci on the real axis are determined by open-loop poles and zeros lying on it. The complex-conjugate poles and complex conjugate zeros of the open-loop transfer function have no effect on the location of the root loci on the real axis.

   In constructing the root loci on the real axis, choose a test point on it. If the **total number of real poles and real zeros** to the **right** of this test point is **odd**, then this point lies on a root locus. If the open-loop poles and open-loop zeros are simple poles and simple zeros, then the root locus and its complement form alternate segments along the real axis.



Root-Locus Plots

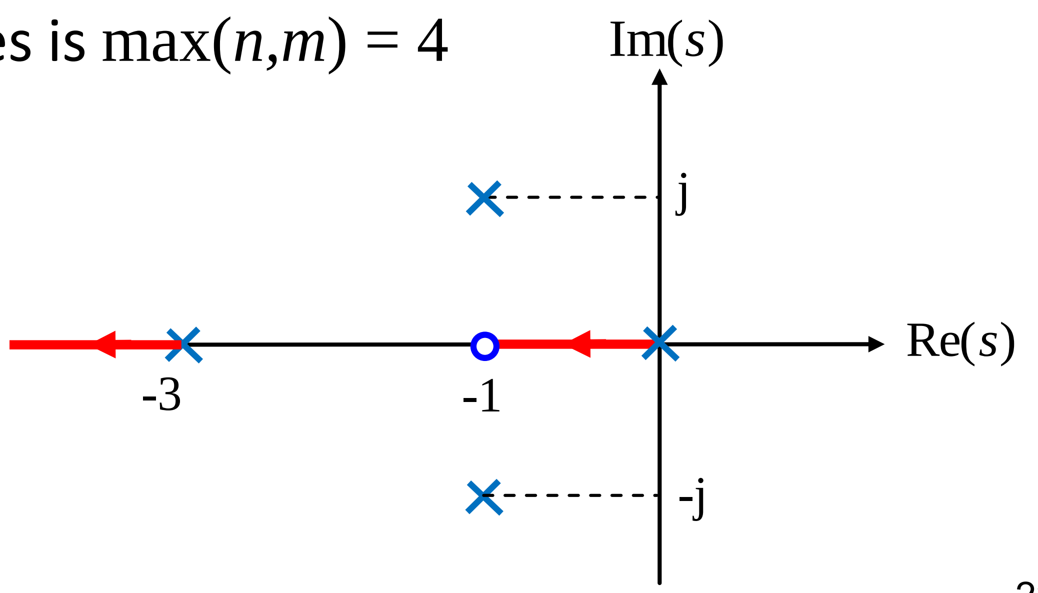
Solution:

4. Determine the **root loci on the real axis**.

Poles are $s_1 = 0$ $s_2 = -3$ $s_{3,4} = -1 \pm j$

Zero is $s_1 = -1$

The number of branches is $\max(n,m) = 4$



Procedure of Root-Locus Plots

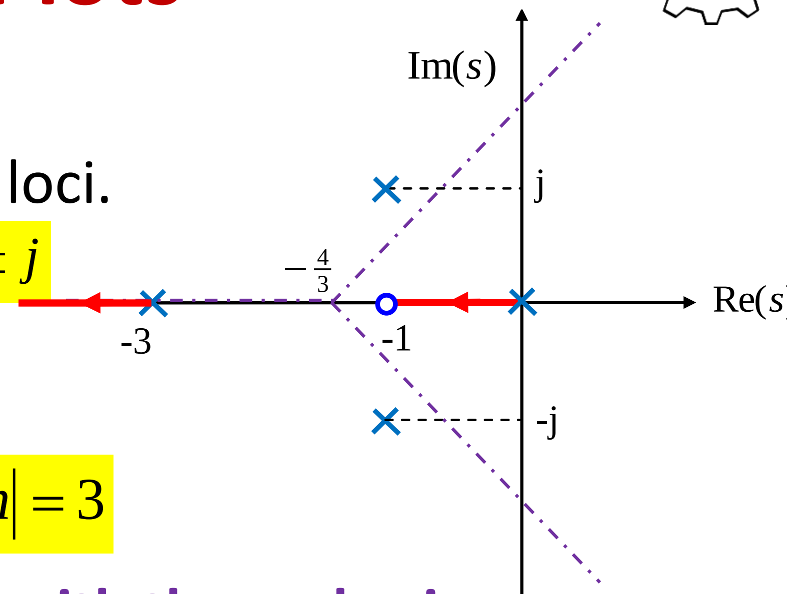
5. Determine the **asymptotes** of root loci. Asymptotes are the lines connecting closed-loop poles to infinite zeros. The number of the asymptotes is $|n-m|$ and their **intersect points with the real axis** are obtained using

$$s = \frac{\sum_{j=1}^{m} z_j - \sum_{j=1}^{n} p_j}{|n-m|}$$

The **angle of asymptotes** with the positive direction of real axis is as follows for $l=0,1,\ldots$

$$\theta_A = \pm \frac{180^\circ (2l+1)}{|n-m|} \qquad K > 0$$

Root-Locus Plots



Solution:

5. Determine the **asymptotes** of root loci.

Poles are $s_1 = 0$ $s_2 = -3$ $s_{3,4} = -1 \pm j$

Zero is $s_1 = -1$

The number of asymptotes is $|n-m| = 3$

The **intersect points of asymptotes with the real axis** are obtained using

$$s = \frac{\sum_{j=1}^{m} z_j - \sum_{j=1}^{n} p_j}{|n-m|} = \frac{(1)-(0+3+1+j+1-j)}{3} = \frac{-4}{3}$$

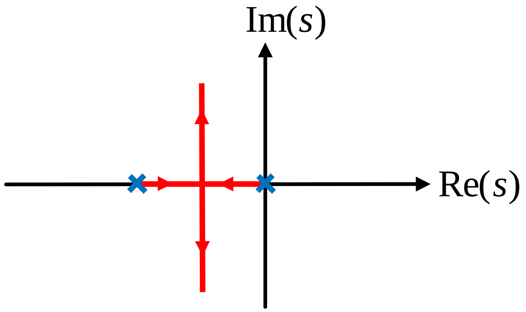
The **angle of asymptotes** with the positive direction of real axis is as follows

$$\theta_A = \pm \frac{\pi(2l+1)}{|n-m|} = \begin{cases} \frac{\pi}{3} \\ \pi \\ -\frac{\pi}{3} \end{cases}$$

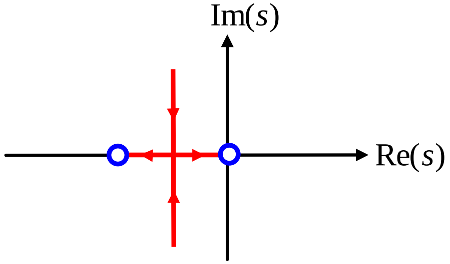
Procedure of Root-Locus Plots

6. Find the **breakaway** and **break-in** points.

If a root locus lies between two adjacent open-loop poles on the real axis, then there exists **at least one breakaway point** between the two poles.



Similarly, if the root locus lies between two adjacent zeros (one zero may be located at minus infinity) on the real axis, then there always exists **at least one break-in point** between the two zeros.



Procedure of Root-Locus Plots

6. Find the **breakaway** and **break-in** points.
   If the root locus lies between an open-loop pole and a zero (finite or infinite) on the real axis, then there may exist **no** breakaway or break-in points **or** there may exist **both** breakaway and break-in points.

   Suppose that the characteristic equation is given by

$$B(s) + KA(s) = 0$$

   The breakaway points and break-in points correspond to **multiple roots** of the characteristic equation. The breakaway and break-in points can be determined from the roots of

$$\frac{dK}{ds} = -\frac{1}{A^2(s)}\left(A(s)\frac{dB(s)}{ds} - B(s)\frac{dA(s)}{ds} \right) = 0$$

Procedure of Root-Locus Plots

6. Find the **breakaway** and **break-in** points.

$$\frac{dK}{ds} = -\frac{1}{A^2(s)}\left(A(s)\frac{dB(s)}{ds} - B(s)\frac{dA(s)}{ds} \right) = 0$$

If a real root of the above equation lies on the root-locus portion of the real axis, then it is an actual breakaway or break-in point, otherwise it is not.

If two roots are a complex-conjugate pair and if it is not certain whether they are on root loci, then it is necessary to check the corresponding K value. If the value of K is positive, that point is an actual breakaway or break-in point.

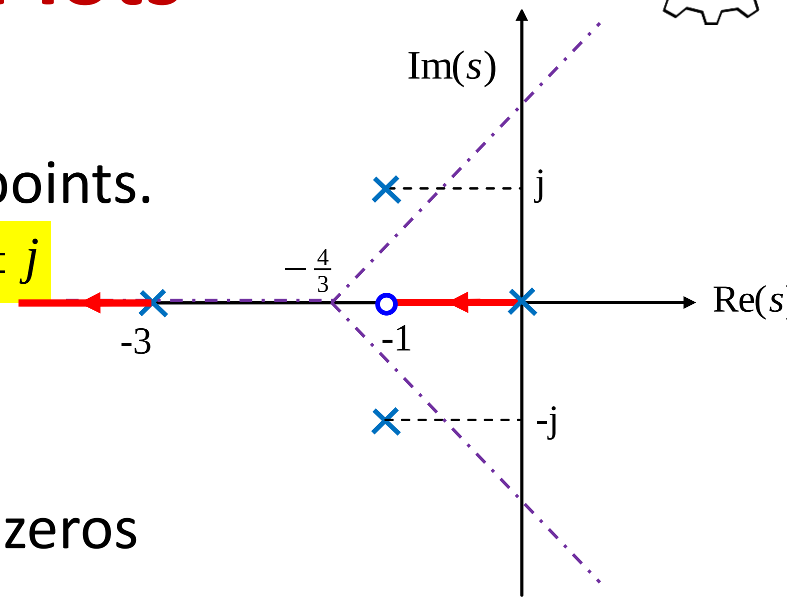
Root-Locus Plots

Solution:

7. Find the **breakaway** and **break-in** points.

Poles are $s_1 = 0$ $s_2 = -3$ $s_{3,4} = -1 \pm j$

Zero is $s_1 = -1$



There is no adjacent poles or adjacent zeros on the real axis

$$\Delta(s) = s(s+3)(s^2+2s+2) + K(s+1) = 0 \quad \Rightarrow \quad K = -\frac{s(s+3)(s^2+2s+2)}{(s+1)}$$

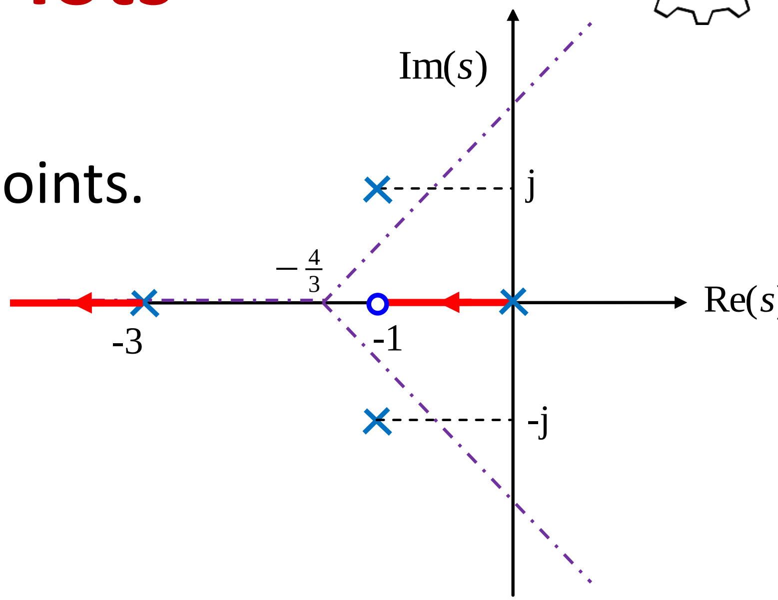
$$\frac{dK}{ds} = -\frac{d}{ds}\left[\frac{s(s+3)(s^2+2s+2)}{(s+1)}\right] = 0 \quad \Rightarrow \quad \frac{dK}{ds} = -\frac{3s^4+14s^3+23s^2+16s+6}{(s+1)^2} = 0$$

$s_1 = -2.63$ $s_2 = 0.26$ $s_{3,4} = -1.15 \pm j1.26$

Root-Locus Plots

Solution:

7. Find the **breakaway** and **break-in** points.



$$K = -\frac{s(s+3)(s^2+2s+2)}{(s+1)}$$

$$\frac{dK}{ds} = -\frac{3s^4+14s^3+23s^2+16s+6}{(s+1)^2} = 0$$

$s_1 = -2.63$ $\quad$ $s_2 = 0.26$ $\quad$ $s_{3,4} = -1.15 \pm j1.26$

The real roots are not on the root locus.

$$K\big|_{s_{3,4}=-1.15\pm j1.26} = -0.45 \pm j2$$

The value of K corresponding to complex roots is not a positive real. **Therefore no break point in this system.**

Procedure of Root-Locus Plots

7. Determine the **angle of departure** (**angle of arrival**) of the root locus **from a complex pole** (**at a complex zero**)

   This angle should satisfy the **angle condition**:

$$\angle N(s) - \angle D(s) = 180^\circ (2l+1)$$

Angle of departure from a complex pole =
180° – (sum of the angles of vectors to the complex pole from other poles) + (sum of the angles of vectors to the complex pole from zeros)

Angle of arrival at a complex zero =
180° – (sum of the angles of vectors to the complex zero from other zeros) + (sum of the angles of vectors to the complex zero from poles)

Root-Locus Plots

Solution:

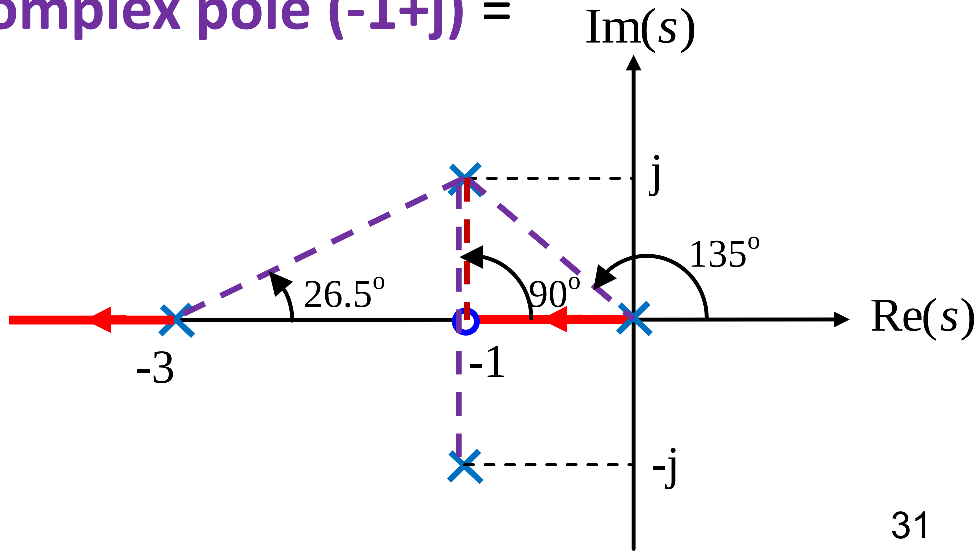
7. Determine the **angle of departure** (**angle of arrival**) of the root locus **from a complex pole** (**at a complex zero**)

   Poles are $s_1 = 0$ $s_2 = -3$ $s_{3,4} = -1 \pm j$

   Zero is $s_1 = -1$

Angle of departure from the complex pole (-1+j) =

180 – (135+90+26.5)+90=18.5°



Procedure of Root-Locus Plots

8. Find the **points** where the root loci may **cross the imaginary axis**.

   The points where the root loci intersect the $j\omega$ axis can be found easily by

   (a) use of Routh's stability criterion or

   (b) letting $s=j\omega$ in the characteristic equation, equating both the real part and the imaginary part to zero, and solving for ω and K.

   The values of ω thus found give the frequencies at which root loci cross the imaginary axis.

   The K value corresponding to each crossing frequency gives the gain at the crossing point.

Root-Locus Plots

Solution:

8. Find the **points** where the root loci may **cross the imaginary axis**.

$$\Delta(s) = s(s+3)(s^2+2s+2) + K(s+1) = 0 \Rightarrow \Delta(s) = s^4 + 5s^3 + 7s^2 + (K+6)s + K = 0$$

Routh's stability criterion:

s^4	1	7	K
s^3	5	$K+6$	0
s^2	$\frac{29-K}{5}$	K	
s^1	$\frac{-K^2-2K+174}{5}$	0	
s^0	K		

Root-Locus Plots

Solution:

8. Find the **points** where the root loci may **cross the imaginary axis**.

To have imaginary roots, one of the entries on the first column should be zero so that the above and below entries do not change sign. At row s^2, $K=29$ but the below entry will be negative.

At row s^1 we have

$$-K^2 - 2K + 174 = 0$$

$$K = 12.23$$

$$K = -14.23$$ Not accepted

s^4	1	7	K
s^3	5	$K+6$	0
s^2	$\frac{29-K}{5}$	K	
s^1	$\frac{-K^2-2K+174}{5}$	0	
s^0	K		

Root-Locus Plots

Solution:

8. Find the **points** where the root loci may **cross the imaginary axis**.

$$K = 12.23$$

The auxiliary polynomial from s^2 row is

$$3.35s^2 + 12.23 = 0$$

The roots of this polynomial are the cross points with imaginary axis:

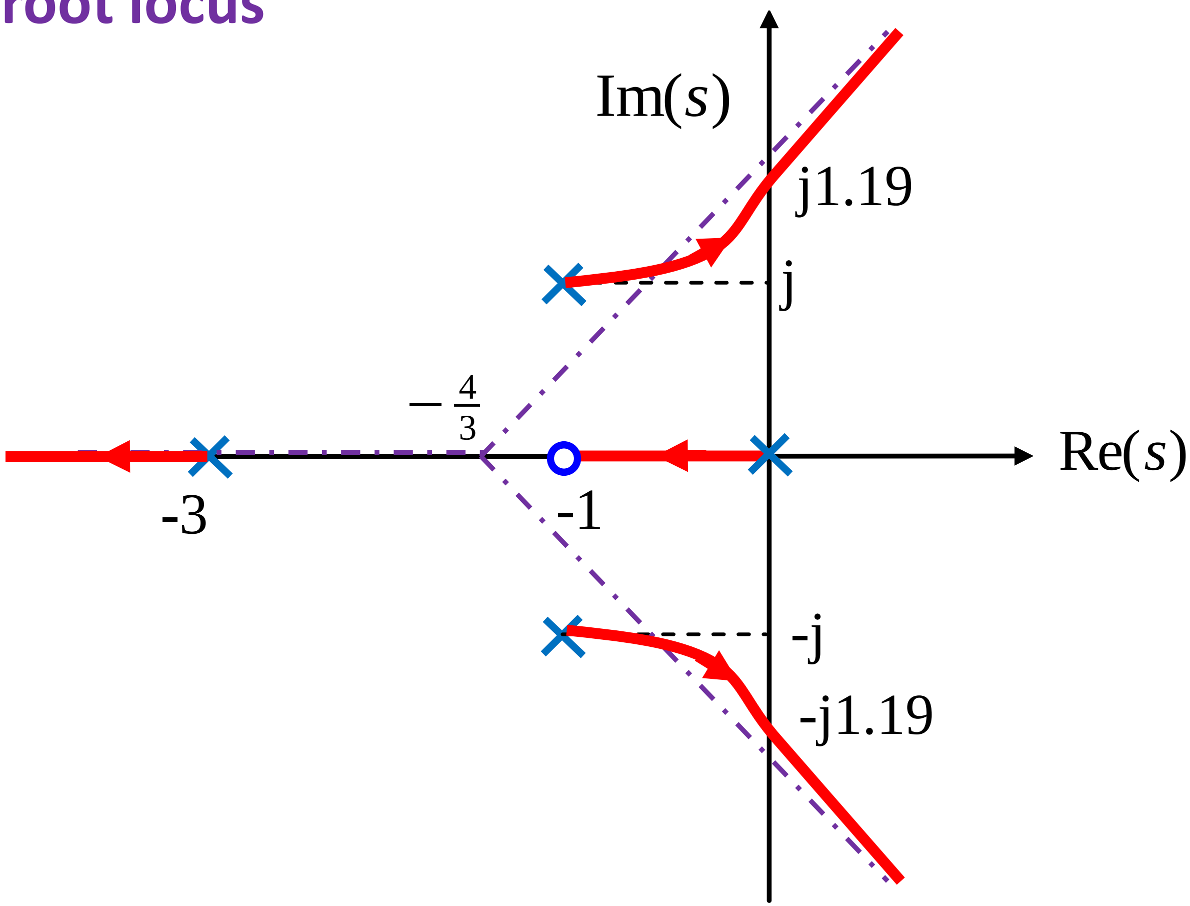
$$s = \pm j1.19$$

s^4	1	7	K
s^3	5	$K+6$	0
s^2	$\frac{29-K}{5}$	K	
s^1	$\frac{-K^2-2K+174}{5}$	0	
s^0	K		

Root-Locus Plots

Solution:

8. Plot the root locus



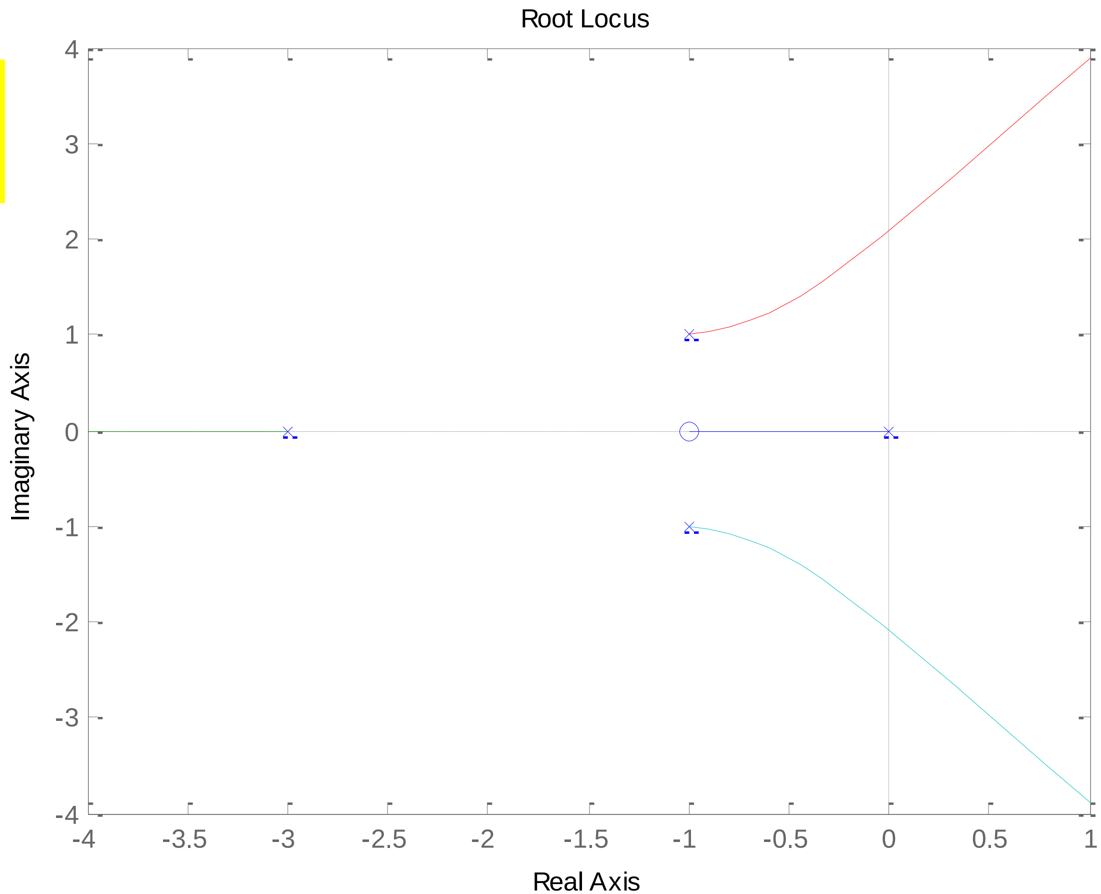
Root-Locus Plots

MATLAB:

$$\frac{N(s)}{D(s)} = \frac{s+1}{s(s+3)(s^2+2s+2)}$$

$$\frac{N(s)}{D(s)} = \frac{s+1}{s^4+5s^3+8s^2+6s}$$

```
>> num=([1 1]);
>> den=([1 5 8 6 0])
>> rlocus(num,den)
```



Procedure of Root-Locus Plots

9. Taking a series of test points in the **broad neighborhood of the origin** of the *s* plane, sketch the root loci.

   Determine the root loci in the broad neighborhood of the $j\omega$ axis and the origin.

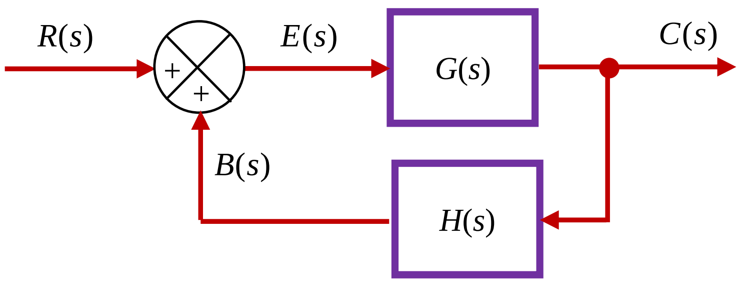
   The most important part of the root loci is on neither the real axis nor the asymptotes but is in the broad neighborhood of the $j\omega$ axis and the origin.

Root-Locus Plots of Positive Feedback Systems

The aim is to find the location of closed-loop poles in s plane with respect to the variation of a parameter K.

Angle and Magnitude Conditions. Consider the positive feedback system shown below. The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1-G(s)H(s)}$$



Root-Locus Plots of Positive Feedback Systems

- The characteristic equation for this closed-loop system is obtained by setting the denominator equal to zero. That is,

$$1 - G(s)H(s) = 0$$

or

$$G(s)H(s) = 1 \qquad (1)$$

- Here we assume that $G(s)H(s)$ is a ratio of polynomials in s.
- Since $G(s)H(s)$ is a complex quantity, Equation (1) can be split into two equations by equating the angles and magnitudes of both sides, respectively, to obtain the following:

Root-Locus Plots of Positive Feedback Systems

$$G(s)H(s) = 1$$

- **Angle condition**: $\angle G(s)H(s) = \pm 180^{\circ}(2l) \quad l = 0, 1, 2, \ldots$

$$\Rightarrow \quad \angle N(s) - \angle D(s) = \pm 180^{\circ}(2l) \quad l = 0, 1, 2, \ldots$$

- **Magnitude condition**:

$$|G(s)H(s)| = 1 \quad \Rightarrow \quad |K| = \left| \frac{D(s)}{N(s)} \right|$$

The values of s that fulfill both the angle and magnitude conditions are the roots of the characteristic equation, or the closed-loop poles.

Modification in Procedure of Root-Locus Plots of Positive Feedback Systems

5. Determine the **asymptotes** of root loci. Asymptotes are the lines connecting closed-lop poles to infinite zeros. The number of the asymptotes is $|n-m|$ and their **intersect points with the real axis** are obtained using

$$s = \frac{\sum_{j=1}^{m} z_j - \sum_{j=1}^{n} p_j}{|n-m|}$$

The **angle of asymptotes** with the positive direction of real axis is as follows for $l=0,1,\ldots$

$$\theta_A = \pm \frac{180^\circ (2l)}{|n-m|}$$

Modification in Procedure of Root-Locus Plots of Positive Feedback Systems

7. Determine the **angle of departure** (**angle of arrival**) of the root locus **from a complex pole** (**at a complex zero**)

   This angle should satisfy the **angle condition**:

$$\angle N(s) - \angle D(s) = 180^{\circ}(2l)$$

Angle of departure from a complex pole =
– (sum of the angles of vectors to the complex pole from other poles) + (sum of the angles of vectors to the complex pole from zeros)

Angle of arrival at a complex zero =
– (sum of the angles of vectors to the complex zero from other zeros) + (sum of the angles of vectors to the complex zero from poles)