In The Name of God The Most Compassionate, The Most Merciful

Linear Control Systems

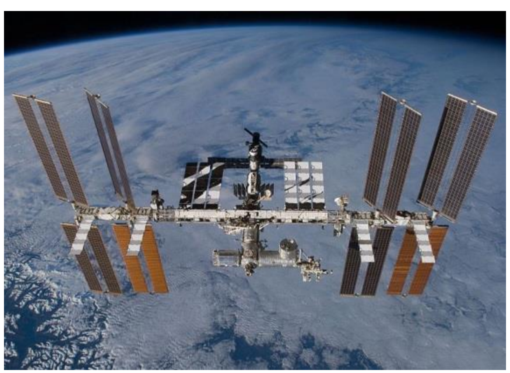


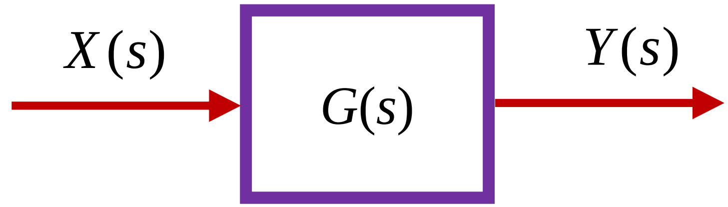
Table of Contents

Chapter 5
Frequency Response Analysis

Introduction

- By the term ***frequency response***, we mean the steady-state response of a system to a **sinusoidal** input.
- In frequency-response methods, we **vary** the **frequency** of the input signal over a certain range and **study** the resulting **response**.
- Assume the following system, if the input is sinusoidal

$$x(t) = A\sin(\omega t)$$



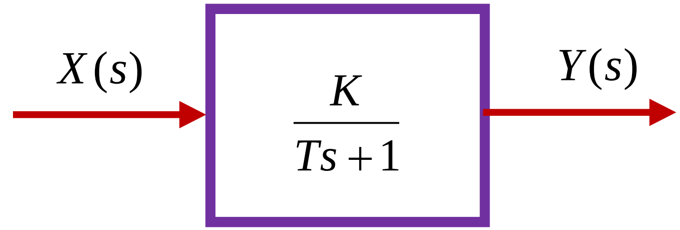
The steady state output is

$$y_{ss}(t) = A|G(j\omega)|\sin(\omega t + \angle G(j\omega))$$

where $G(j\omega)$ is called the sinusoidal transfer function.

Introduction

- **Example 1**: Find the steady-state output of the following system in response to $x(t) = A\sin(\omega t)$



$$G(s) = \frac{K}{Ts+1}$$

$$G(j\omega) = \frac{K}{jT\omega + 1} \quad \begin{cases} |G(j\omega)| = \dfrac{K}{\sqrt{1+T^2\omega^2}} \\ \angle G(j\omega) = -\tan^{-1} T\omega \end{cases}$$

$$y_{ss}(t) = A|G(j\omega)|\sin(\omega t + \angle G(j\omega))$$

$$y_{ss}(t) = \frac{AK}{\sqrt{1+T^2\omega^2}}\sin(\omega t - \tan^{-1} T\omega)$$

Presenting Frequency-Response Characteristics in Graphical Forms

- The **sinusoidal transfer function**, a complex function of the frequency ω, is characterized by its **magnitude** and **phase** angle, with **frequency** as the parameter.

- There are **three commonly used representations** of **sinusoidal transfer functions**:

1. **Bode diagram or logarithmic plot**

$$|G(j\omega)| \ vs. \ \omega$$
$$\angle G(j\omega) \ vs. \ \omega$$

2. **Nyquist plot or polar plot**

$$\mathrm{Im}[G(j\omega)] \ vs. \ \mathrm{Re}[G(j\omega)]$$

3. **Log-magnitude-versus-phase plot (Nichols plots)**

$$|G(j\omega)| \ vs. \ \angle G(j\omega)$$

Bode Diagrams

A **Bode diagram** consists of two graphs:

1. One is a plot of the **logarithm of the magnitude** of a sinusoidal transfer function;
2. The other is a plot of the **phase angle**; Both are plotted against the frequency on a logarithmic scale.

The standard representation of the logarithmic magnitude of $G(j\omega)$ is $20 \log |G(j\omega)|$, where the base of the logarithm is 10. The unit used in this representation of the magnitude is the decibel (dB).

Basic Factors

The **basic factors** that very frequently occur in an arbitrary transfer function $G(j\omega)H(j\omega)$ are

1. **Gain** K

2. **Integral** and **derivative** factors $(j\omega)^{\pm 1}$

3. **First-order** factors $(1+j\omega)^{\pm 1}$

4. **Quadratic** factors $\left[1+2\zeta(j\omega/\omega_n)+(j\omega/\omega_n)^2\right]^{\pm 1}$

Note that **adding the logarithms** of the gains corresponds to **multiplying** them together.

Basic Factors

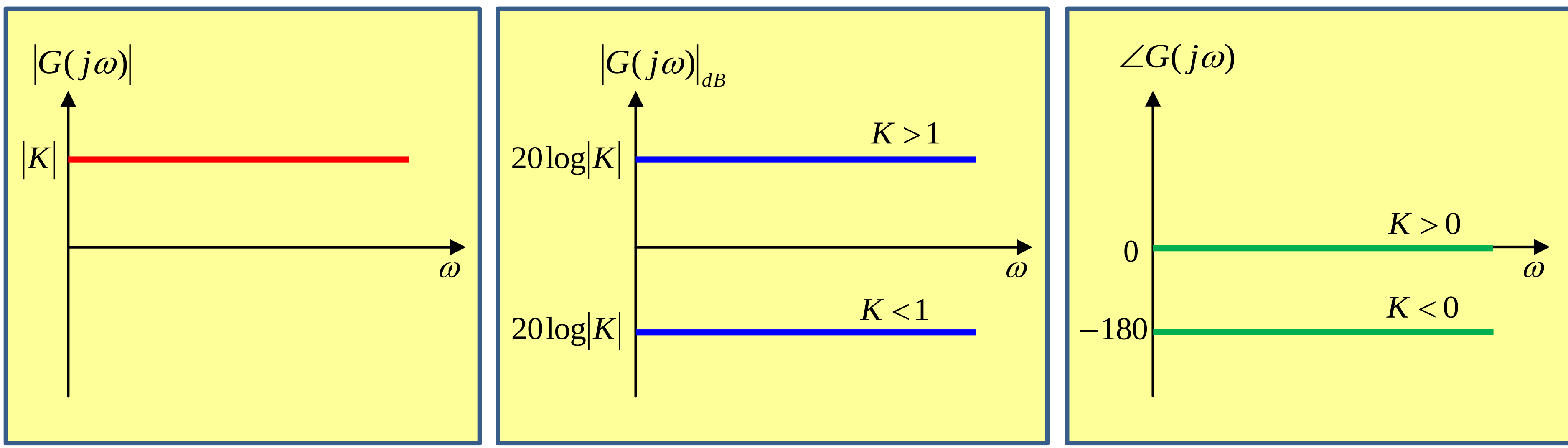
1. Gain K

$$G(s)=K \Rightarrow G(j\omega)=K$$

$$|G(j\omega)|=|K|$$

$$|G(j\omega)|_{dB}=20\log|K|$$

$$\angle G(j\omega)=\begin{cases}0 & K\geq 0\\ -180^{\circ} & K<0\end{cases}$$



Basic Factors

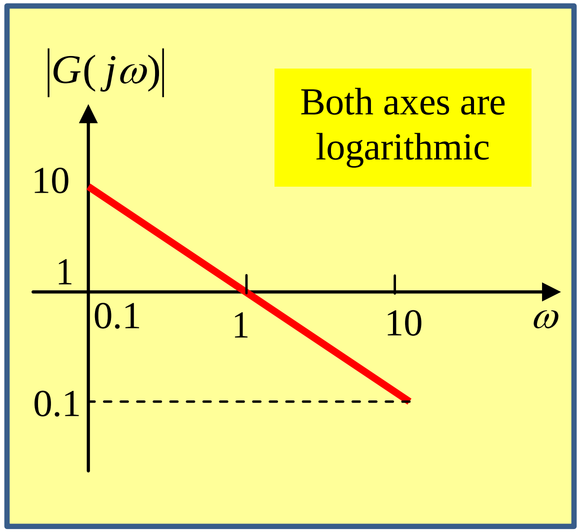
2. Integral $(j\omega)^{-1}$

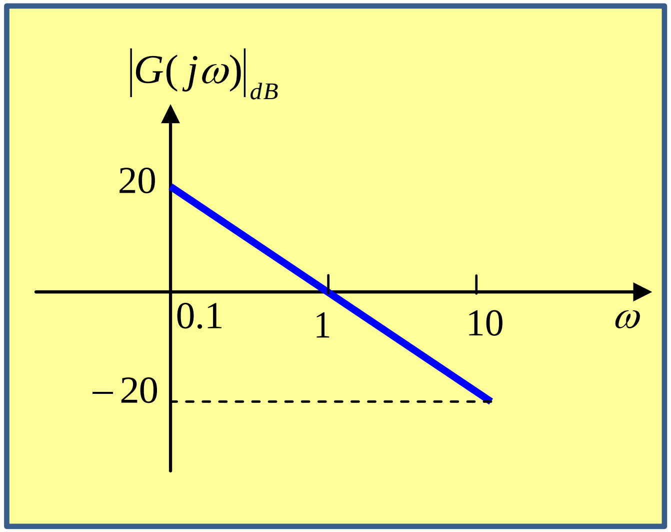
$$G(s)=\frac{1}{s} \;\Rightarrow\; G(j\omega)=\frac{1}{j\omega}$$

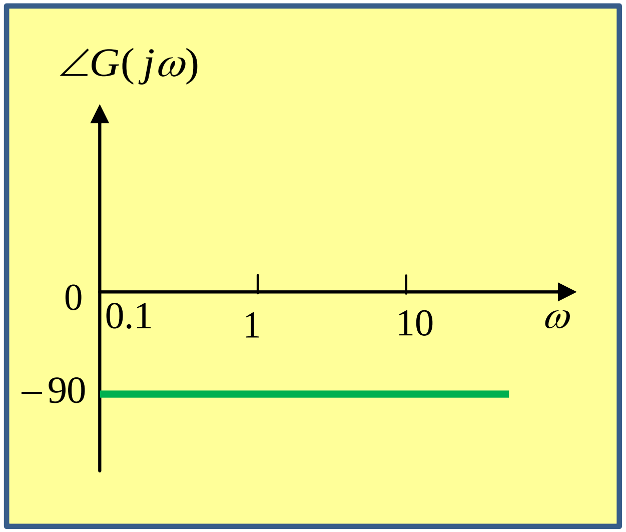
$$\left|G(j\omega)\right|=\frac{1}{\omega}$$

$$\left|G(j\omega)\right|_{dB}=-20\log\omega$$

$$\angle G(j\omega)=-90^{\circ}$$







Basic Factors

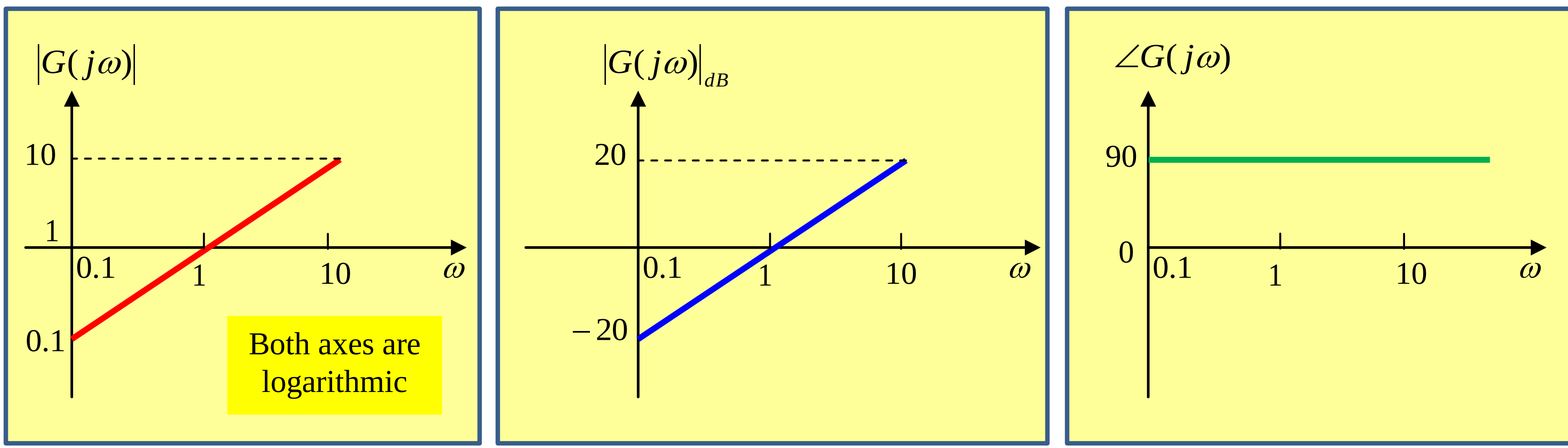
3. Derivative $(j\omega)$

$G(s) = s$ ⇒ $G(j\omega) = j\omega$

$$|G(j\omega)| = \omega$$

$$|G(j\omega)|_{dB} = 20\log\omega$$

$$\angle G(j\omega) = +90^\circ$$



Basic Factors

4. First-order $(1+j\omega T)^{-1}$

$$G(s)=\frac{1}{1+Ts} \Rightarrow G(j\omega)=\frac{1}{1+jT\omega}$$

$$\left|G(j\omega)\right|=\frac{1}{\sqrt{1+T^2\omega^2}} \quad (1)$$

$$\angle G(j\omega)=-\tan^{-1}(T\omega) \quad (2)$$

(1) $\Rightarrow$ $$\left|G(j\omega)\right|=\begin{cases}\dfrac{1}{T\omega} & \omega>>\frac{1}{T}\\ 1 & \omega<<\frac{1}{T}\end{cases} \Rightarrow \left|G(j\omega)\right|_{dB}=\begin{cases}-20\log T\omega & \omega>>\frac{1}{T}\\ 0 & \omega<<\frac{1}{T}\end{cases}$$

$$\left|G(j\omega)\right|_{\omega=\frac{1}{T}}=\frac{1}{\sqrt{2}}$$

$$\left|G(j\omega)\right|_{dB\ \ \omega=\frac{1}{T}}=-20\log\sqrt{2}=-3\ \ dB$$

Basic Factors

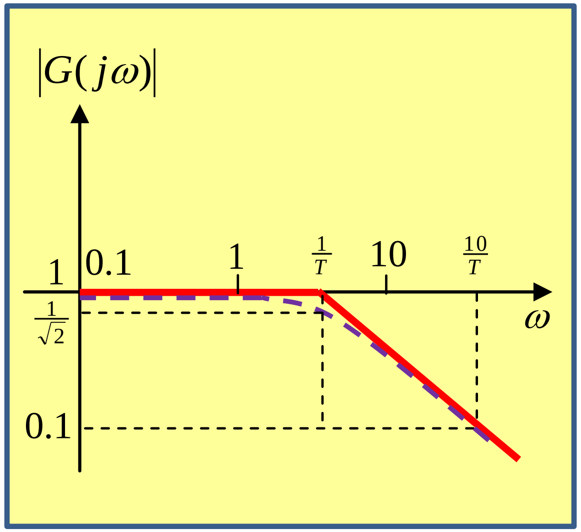
4. First-order $(1+j\omega T)^{-1}$

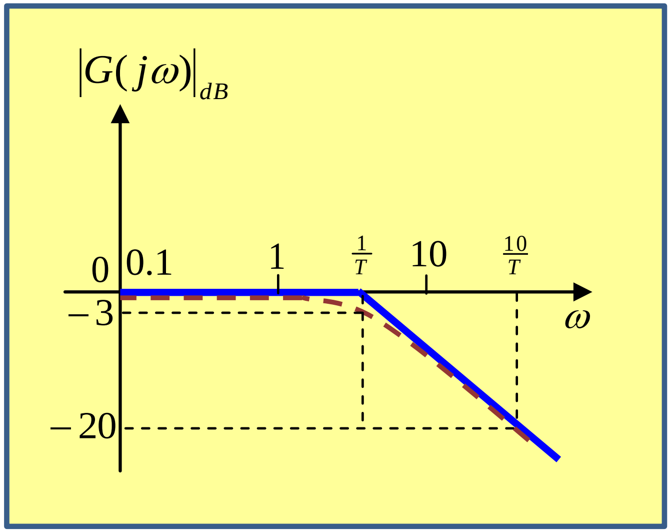
$$G(s)=\frac{1}{1+Ts}$$

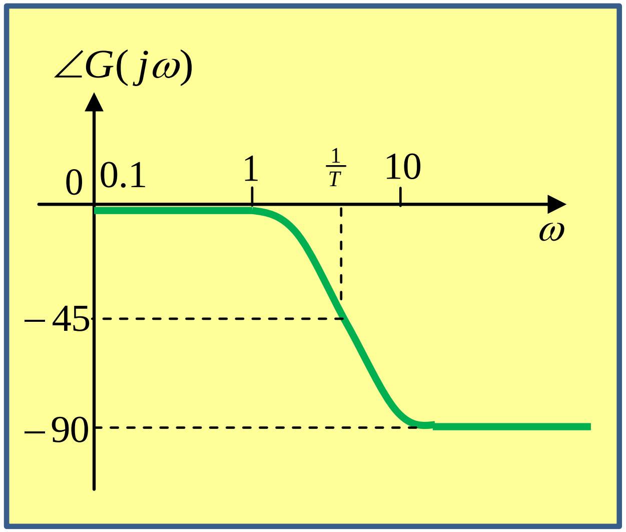
$$|G(j\omega)|=\begin{cases}\dfrac{1}{T\omega} & \omega>>\frac{1}{T}\\ 1 & \omega<<\frac{1}{T}\end{cases}$$

$$\angle G(j\omega)=-\tan^{-1}(T\omega)$$

$$|G(j\omega)|_{dB}=\begin{cases}-20\log T\omega & \omega>>\frac{1}{T}\\ 0 & \omega<<\frac{1}{T}\end{cases}$$







Basic Factors

5. First-order $(1+j\omega T)$

$$G(s)=1+Ts \quad \Rightarrow \quad G(j\omega)=1+jT\omega$$

$$|G(j\omega)|=\sqrt{1+T^2\omega^2} \qquad (1)$$

$$\angle G(j\omega)=\tan^{-1}(T\omega) \qquad (2)$$

(1) $\Rightarrow$ $$|G(j\omega)|=\begin{cases} T\omega & \omega >> \frac{1}{T} \\ 1 & \omega << \frac{1}{T} \end{cases} \quad \Rightarrow \quad |G(j\omega)|_{dB}=\begin{cases} 20\log T\omega & \omega >> \frac{1}{T} \\ 0 & \omega << \frac{1}{T} \end{cases}$$

$$|G(j\omega)|_{\omega=\frac{1}{T}}=\sqrt{2}$$

$$|G(j\omega)|_{dB \quad \omega=\frac{1}{T}}=20\log\sqrt{2}=3 \ dB$$

Basic Factors

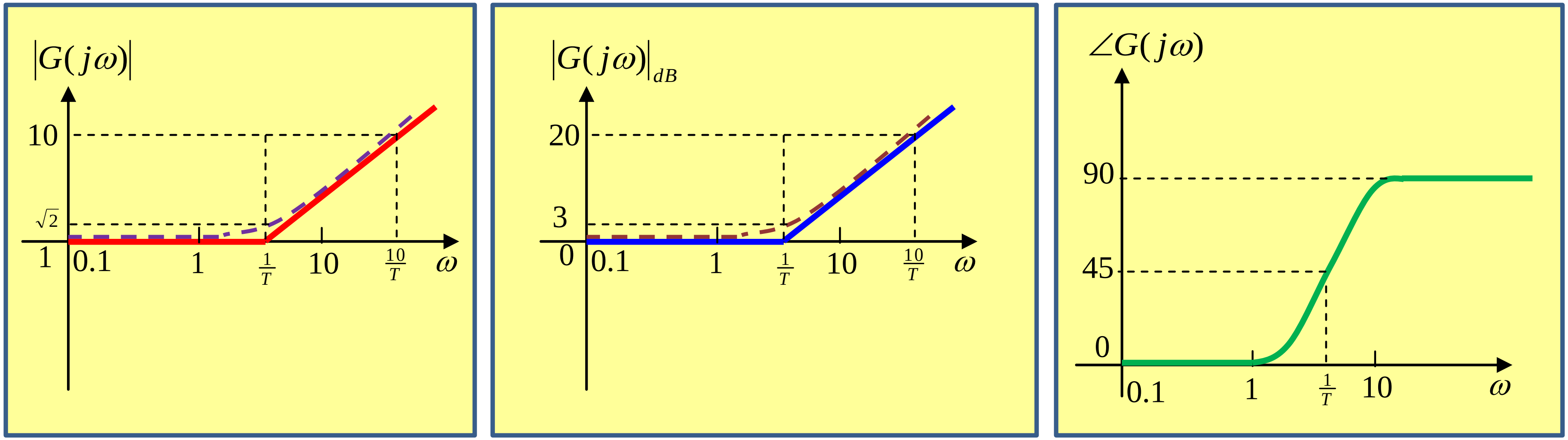
5. First-order $(1+j\omega T)$

$G(s)=1+Ts$

$$|G(j\omega)|=\begin{cases}T\omega & \omega>>\frac{1}{T}\\ 1 & \omega<<\frac{1}{T}\end{cases}$$

$$\angle G(j\omega)=\tan^{-1}(T\omega)$$

$$|G(j\omega)|_{dB}=\begin{cases}20\log T\omega & \omega>>\frac{1}{T}\\ 0 & \omega<<\frac{1}{T}\end{cases}$$



Basic Factors

6. First-order $(-1+j\omega T)$

$G(s)=-1+Ts$ → $G(j\omega)=-1+jT\omega$

$$|G(j\omega)|=\sqrt{1+T^2\omega^2} \quad (1)$$

$$\angle G(j\omega)=\tan^{-1}\left(\tfrac{T\omega}{-1}\right) \quad (2)$$

(1) → $$|G(j\omega)|=\begin{cases} T\omega & \omega >> \frac{1}{T} \\ 1 & \omega << \frac{1}{T} \end{cases}$$ → $$|G(j\omega)|_{dB}=\begin{cases} 20\log T\omega & \omega >> \frac{1}{T} \\ 0 & \omega << \frac{1}{T} \end{cases}$$

$$|G(j\omega)|_{\omega=\frac{1}{T}}=\sqrt{2}$$

$$|G(j\omega)|_{dB\ \ \omega=\frac{1}{T}}=20\log\sqrt{2}=3\ \ dB$$

Basic Factors

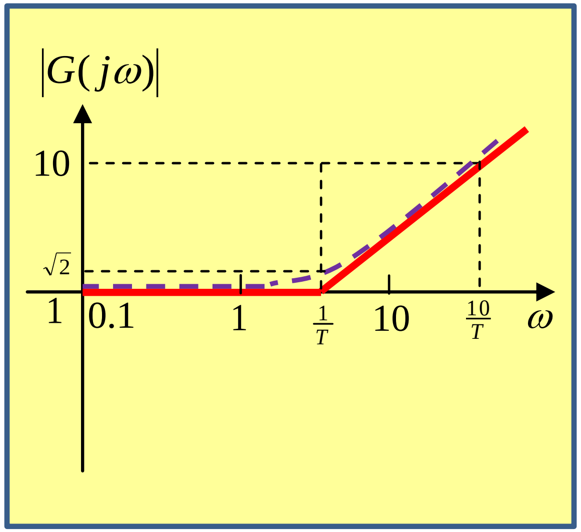
6. First-order $(-1+j\omega T)$

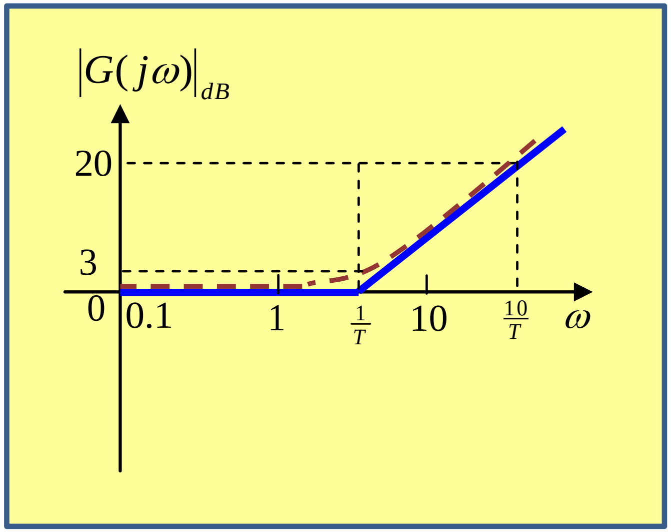
$G(s)=-1+Ts$

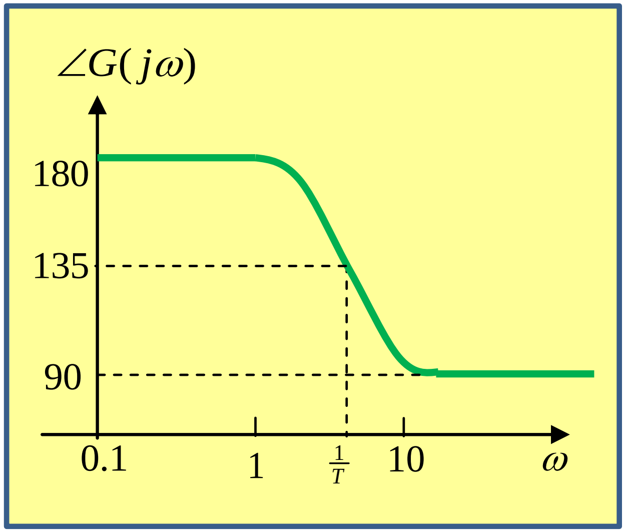
$$|G(j\omega)|=\begin{cases} T\omega & \omega>>\frac{1}{T} \\ 1 & \omega<<\frac{1}{T}\end{cases}$$

$$\angle G(j\omega)=\tan^{-1}(\tfrac{T\omega}{-1})$$

$$|G(j\omega)|_{dB}=\begin{cases} 20\log T\omega & \omega>>\frac{1}{T} \\ 0 & \omega<<\frac{1}{T}\end{cases}$$







Basic Factors

7. Second-order

$$G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$G(s)=\frac{1}{\dfrac{s^2}{\omega_n^2}+\dfrac{2\zeta}{\omega_n}s+1}$$

$$G(j\omega)=\frac{1}{\left(1-\dfrac{\omega^2}{\omega_n^2}\right)+j2\zeta\dfrac{\omega}{\omega_n}}$$

$$\left|G(j\omega)\right|=\frac{1}{\sqrt{\left(1-\dfrac{\omega^2}{\omega_n^2}\right)^2+\left(2\zeta\dfrac{\omega}{\omega_n}\right)^2}}$$

$$\angle G(j\omega)=-\tan^{-1}\left(\frac{2\zeta\frac{\omega}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}}\right)$$

Basic Factors

7. Second-order

$$G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$|G(j\omega)|=\frac{1}{\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(2\zeta\frac{\omega}{\omega_n}\right)^2}}$$

$$\angle G(j\omega)=-\tan^{-1}\left(\frac{2\zeta\frac{\omega}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}}\right)$$

$$|G(j\omega)|=\begin{cases}\left(\frac{\omega}{\omega_n}\right)^{-2} & \omega>>\omega_n\\ 1 & \omega<<\omega_n\end{cases}$$

$$|G(j\omega)|_{dB}=\begin{cases}-40\log\frac{\omega}{\omega_n} & \omega>>\omega_n\\ 0 & \omega<<\omega_n\end{cases}$$

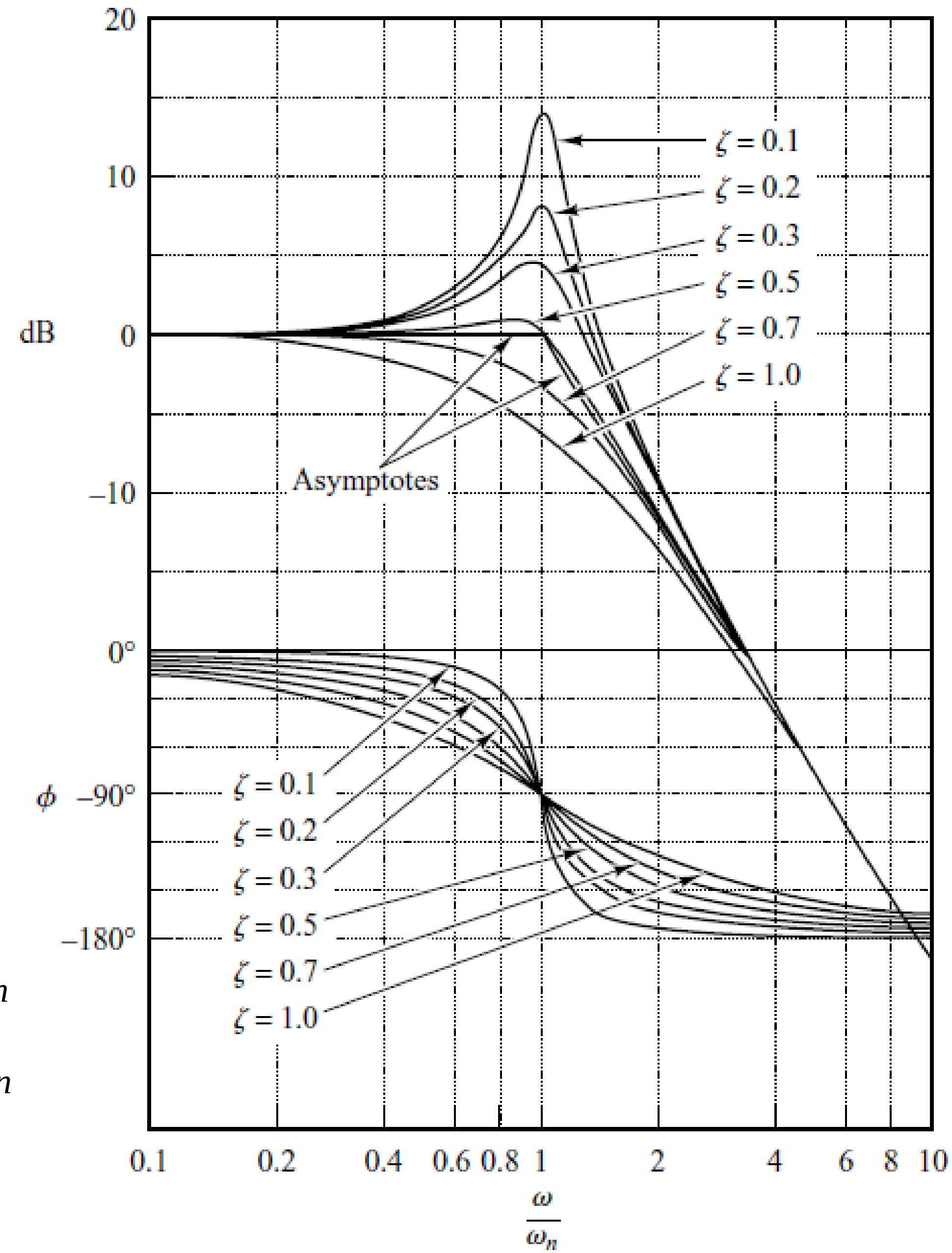
Basic Factors

7. Second-order

$$G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$\angle G(j\omega)=-\tan^{-1}\left(\frac{2\zeta\frac{\omega}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}}\right)$$

$$\left|G(j\omega)\right|_{dB}=\begin{cases}-40\log\frac{\omega}{\omega_n} & \omega>>\omega_n\\ 0 & \omega<<\omega_n\end{cases}$$



Basic Factors

7. Second-order

The Resonant Frequency ω_r and the Resonant Peak Value M_r

The peak value of $|G(j\omega)|$ occurs when the denominator, $g(\omega)$, minimizes

$$g(\omega) = \sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta \frac{\omega}{\omega_n}\right)^2}$$

$$\frac{dg(\omega)}{d\omega} = 0 \quad \Rightarrow \quad \omega_r = \omega_n \sqrt{1 - 2\zeta^2} \qquad 0 \le \zeta < \tfrac{1}{\sqrt{2}}$$

$$M_r = |G(j\omega)|_{\max} = |G(j\omega_r)| = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \qquad 0 \le \zeta < \tfrac{1}{\sqrt{2}}$$

Basic Factors

Corner frequency

- In the **first-order** system of the following form,

$$G(s) = \frac{K}{Ts+1}$$

the corner frequency is $\omega_c = \frac{1}{T}$

- In the **second-order** system of the following form

$$G(s) = \frac{K}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

the corner frequency is $\omega_c = \omega_n$

Basic Factors

Corner frequency

- **Example 1**:

$$G(s) = \frac{4}{3s+2} \quad \Rightarrow \quad G(s) = \frac{2}{\frac{3}{2}s+1}$$

the corner frequency is $\omega_c = \frac{1}{T} = \frac{2}{3}$

- **Example 2**:

$$G(s) = \frac{6}{2s^2+2s+4} \qquad G(s) = \frac{3}{s^2+s+2}$$

the corner frequency is $\omega_c = \omega_n = \sqrt{2}$

Bode Diagrams

Example: Plot the bode diagrams of the following system

$$G(s)=\frac{1000}{s(s+5)(s+50)} \quad \Rightarrow \quad G(j\omega)=\frac{1000}{j\omega(j\omega+5)(j\omega+50)}$$

$$\Rightarrow \quad G(j\omega)=\frac{4}{j\omega(j\frac{\omega}{5}+1)(j\frac{\omega}{50}+1)}$$

The **corner frequencies** are $\omega_{c1}=\frac{1}{T_1}=5$ and $\omega_{c2}=\frac{1}{T_2}=50$

$$\left|G(j\omega)\right|_{dB} = 20\log 4 + 20\log\left|\frac{1}{j\omega}\right| + 20\log\left|\frac{1}{j\frac{\omega}{5}+1}\right| + 20\log\left|\frac{1}{j\frac{\omega}{50}+1}\right|$$

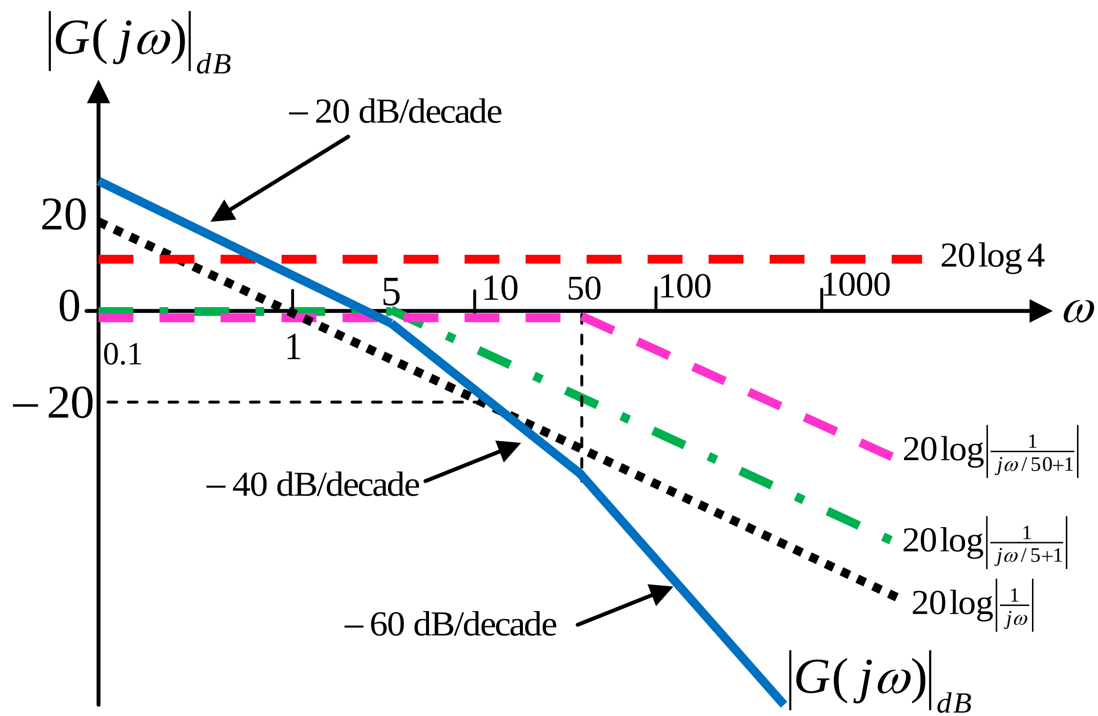
$$\angle G(j\omega)= 0 + \angle\frac{1}{j\omega} + \angle\frac{1}{j\frac{\omega}{5}+1} + \angle\frac{1}{j\frac{\omega}{50}+1}$$

Bode Diagrams (Magnitude)

Example: Plot the bode diagrams of the following system

The **corner frequencies** are $\omega_{c1} = \frac{1}{T_1} = 5$ and $\omega_{c2} = \frac{1}{T_2} = 50$

$$\left|G(j\omega)\right|_{dB} = 20\log 4 + 20\log\left|\frac{1}{j\omega}\right| + 20\log\left|\frac{1}{j\frac{\omega}{5}+1}\right| + 20\log\left|\frac{1}{j\frac{\omega}{50}+1}\right|$$



$$G(j\omega) = \frac{4}{j\omega(j\frac{\omega}{5}+1)(j\frac{\omega}{50}+1)}$$

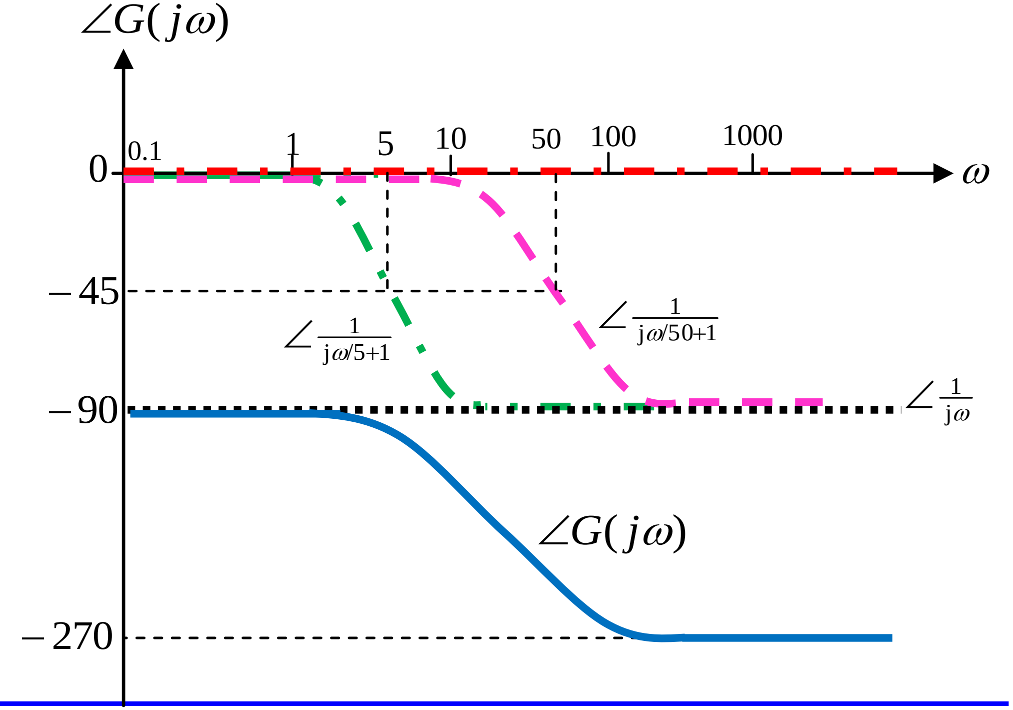
Bode Diagrams (Phase)

Example: Plot the bode diagrams of the following system

The **corner frequencies** are $\omega_{c1} = \frac{1}{T_1} = 5$ and $\omega_{c2} = \frac{1}{T_2} = 50$

$$\angle G(j\omega) = 0 + \angle\frac{1}{j\omega} + \angle\frac{1}{j\frac{\omega}{5}+1} + \angle\frac{1}{j\frac{\omega}{50}+1}$$

$$G(j\omega) = \frac{4}{j\omega(j\frac{\omega}{5}+1)(j\frac{\omega}{50}+1)}$$



Bode Diagrams Using MATLAB

Example: Using MATLAB Plot the bode diagrams of the following system

$$G(s) = \frac{1000}{s(s+5)(s+50)} \Rightarrow G(j\omega) = \frac{4}{j\omega(j\frac{\omega}{5}+1)(j\frac{\omega}{50}+1)}$$

```
w=logspace(-1,3,100);
numT=1000;
denT=[1 55 250 0];
 num1=4;
den1=1;
 num2=1;
den2=[1 0];
 num3=1;
den3=[1/5 1];
```

Bode Diagrams Using MATLAB

Example: Using MATLAB Plot the bode diagrams of the following system

$$G(s) = \frac{1000}{s(s+5)(s+50)} \Rightarrow G(j\omega) = \frac{4}{j\omega(j\frac{\omega}{5}+1)(j\frac{\omega}{50}+1)}$$

```
num4=1;
den4=[1/50 1];

[mag,phase]=bode(numT,denT,w);
[mag1,phase1]=bode(num1,den1,w);
[mag2,phase2]=bode(num2,den2,w);
[mag3,phase3]=bode(num3,den3,w);
[mag4,phase4]=bode(num4,den4,w);
```

Bode Diagrams Using MATLAB

Example: Using MATLAB Plot the bode diagrams of the following system

$$G(s) = \frac{1000}{s(s+5)(s+50)} \Rightarrow G(j\omega) = \frac{4}{j\omega(j\frac{\omega}{5}+1)(j\frac{\omega}{50}+1)}$$

```
figure(1)
loglog(w,mag,'b','linewidth',3)
hold on
loglog(w,mag1,'r--','linewidth',2)
loglog(w,mag2,'k:','linewidth',2)
loglog(w,mag3,'g-.','linewidth',2)
loglog(w,mag4,'m--','linewidth',2)
legend('G(j\omega)','K=4','1/j\omega','1/(j\omega/5+1)','1/(j\omega/50+1)')
```

Bode Diagrams Using MATLAB

Example: Using MATLAB Plot the bode diagrams of the following system

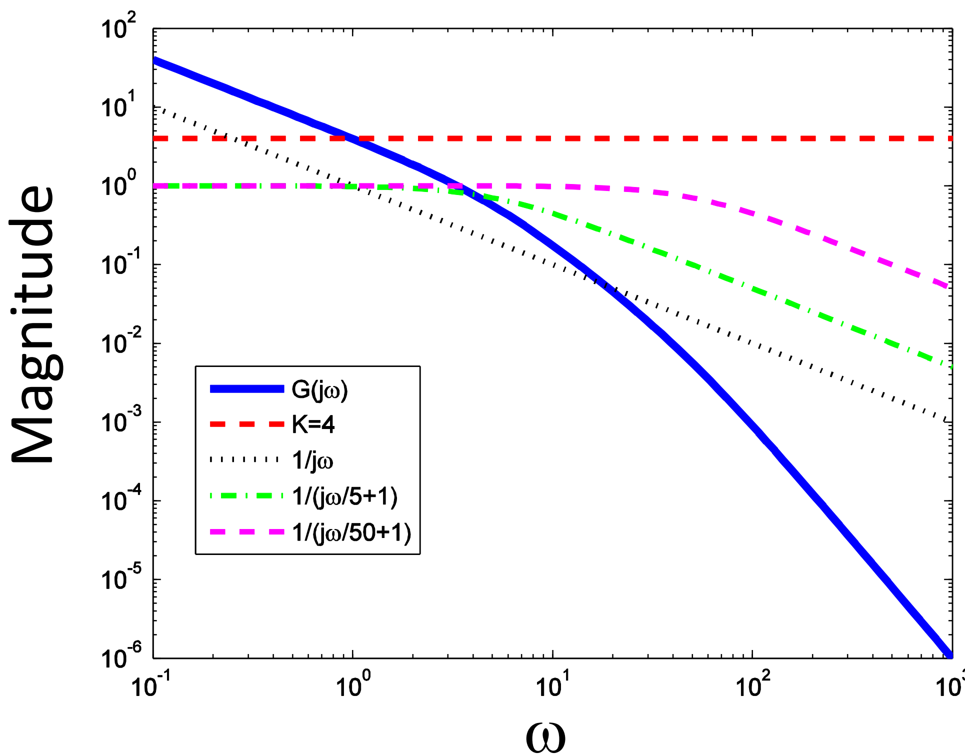
$$G(s)=\frac{1000}{s(s+5)(s+50)} \Rightarrow G(j\omega)=\frac{4}{j\omega(j\frac{\omega}{5}+1)(j\frac{\omega}{50}+1)}$$

```
figure(2)
semilogx(w,phase,'b','linewidth',3)
hold on
semilogx(w,phase1,'r--','linewidth',2)
semilogx(w,phase2,'k:','linewidth',2)
semilogx(w,phase3,'g-.','linewidth',2)
semilogx(w,phase4,'m--','linewidth',2)
legend('G(j\omega)','K=4','1/j\omega','1/(j\omega/5+1)','1/(j\omega
/50+1)')
```

Bode Diagrams Using MATLAB

Example: Using MATLAB Plot the bode diagrams of the following system

$$G(s)=\frac{1000}{s(s+5)(s+50)} \Rightarrow G(j\omega)=\frac{4}{j\omega\left(j\frac{\omega}{5}+1\right)\left(j\frac{\omega}{50}+1\right)}$$



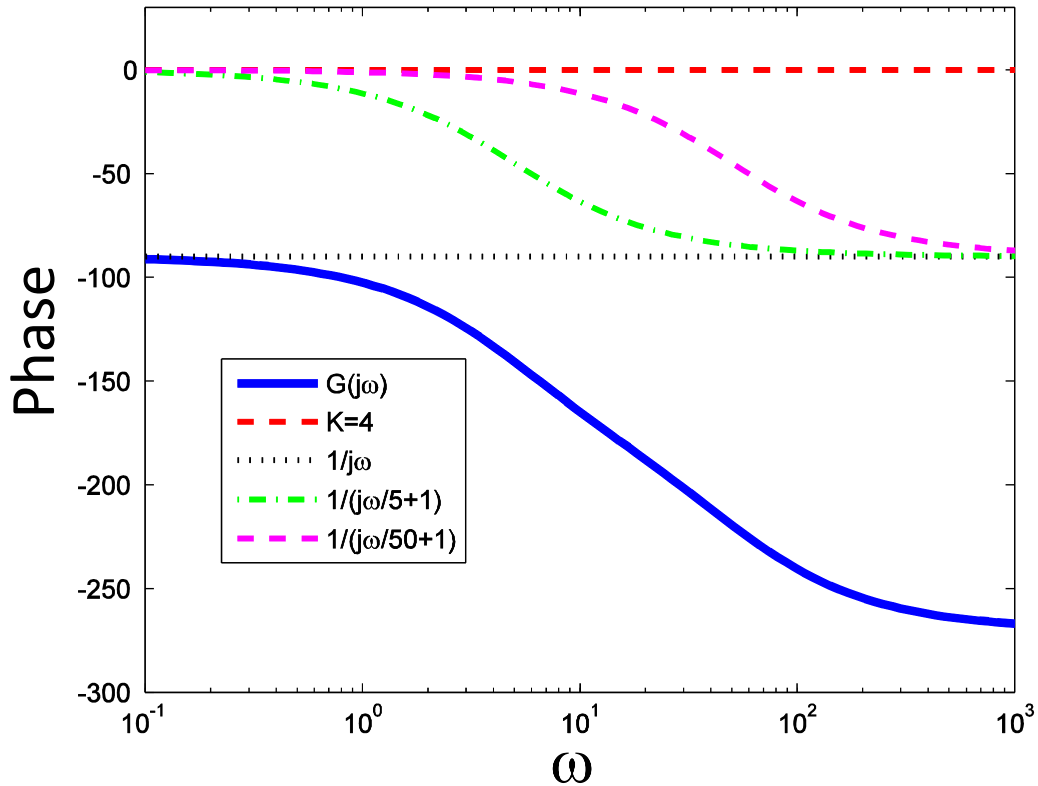
Note that the magnitude is in logarithmic scale.

Magnitude in NOT in dB

Bode Diagrams Using MATLAB

Example: Using MATLAB Plot the bode diagrams of the following system

$$G(s)=\frac{1000}{s(s+5)(s+50)} \Rightarrow G(j\omega)=\frac{4}{j\omega\left(j\frac{\omega}{5}+1\right)\left(j\frac{\omega}{50}+1\right)}$$



Minimum-Phase Systems and Nonminimum-Phase Systems

- Transfer functions having **neither poles nor zeros** in the **right-half** *s* plane are **minimum-phase** transfer functions,

- Whereas those having **poles and/or zeros** in the **right-half** s plane are **nonminimum-phase** transfer functions.

- Systems with **minimum-phase** transfer functions are called ***minimum-phase*** systems,

- whereas those with **nonminimum-phase** transfer functions are called ***nonminimum-phase*** systems.

Transport Lag

- **Transport lag**, which is also called **dead time**, is of non-minimumphase behavior and has an excessive phase lag with no attenuation at high frequencies.

- Such transport lags normally exist in **thermal**, **hydraulic**, and **pneumatic** systems.

- Consider the transport lag given by $G(j\omega) = e^{-j\omega T}$

- The magnitude is always equal to unity, since

$$|G(j\omega)| = |\cos \omega T - j \sin \omega T| = 1 \quad \Rightarrow \quad |G(j\omega)|_{dB} = 0$$

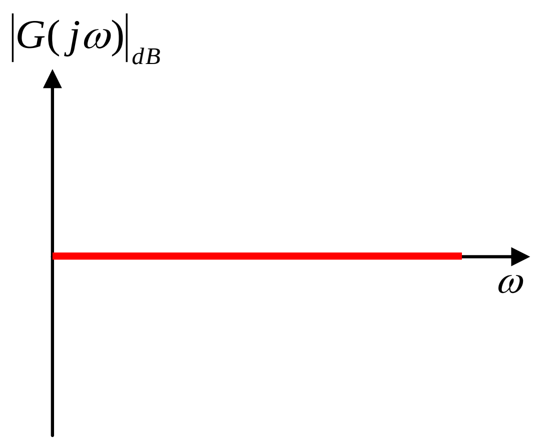
Transport Lag

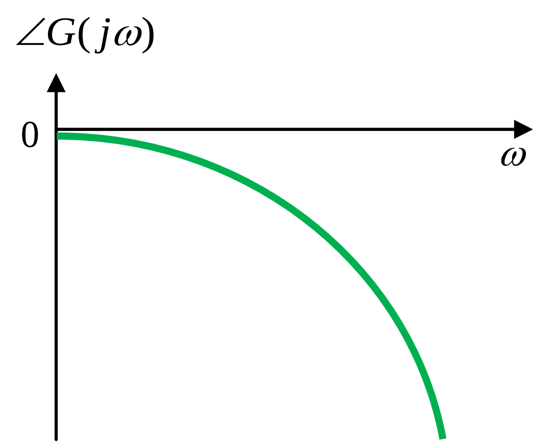
- Consider the transport lag given by $G(j\omega) = e^{-j\omega T}$
- The **magnitude** is always equal to unity, since

$$|G(j\omega)| = |\cos\omega T - j\sin\omega T| = 1 \quad \Rightarrow \quad |G(j\omega)|_{dB} = 0$$

- The **phase** angle is

$$\angle G(j\omega) = -\omega T \quad \text{(radians)} \qquad \angle G(j\omega) = -57.3\omega T \quad \text{(degrees)}$$





Bode Diagrams

- Consider the following system

$$G(s) = \frac{(T_a s+1)(T_b s+1)\cdots(T_m s+1)}{s^N (T_1 s+1)(T_2 s+1)\cdots(T_{n-N} s+1)}$$

- Where the system is of type ***N***, the order of the numerator is ***m*** and the order of the denominator is ***n***.

- The relation between the **start- and end-slopes** of the magnitude Bode diagrams with the **system-Type and order** are as follows

$$\text{Start slope} = -20N \quad \text{dB/decade}$$

$$\text{End slope} = -20(n-m) \quad \text{dB/decade}$$

Bode Diagrams

- Consider the following **minimum-phase** system

$$G(s) = \frac{(T_a s+1)(T_b s+1)\cdots(T_m s+1)}{s^N (T_1 s+1)(T_2 s+1)\cdots(T_{n-N} s+1)}$$

- Where the system is of type N, the order of the numerator is m and the order of the denominator is n.

- The relations between the **start- and end-phase** of the phase Bode diagrams with the **system-Type and order in minimum-phase systems** are as follows

$$\text{Start phase} = -90N \quad \text{degrees}$$

$$\text{End phase} = -90(n-m) \quad \text{degrees}$$

ONLY for minimum-phase systems

Polar Plots (Nyquist)

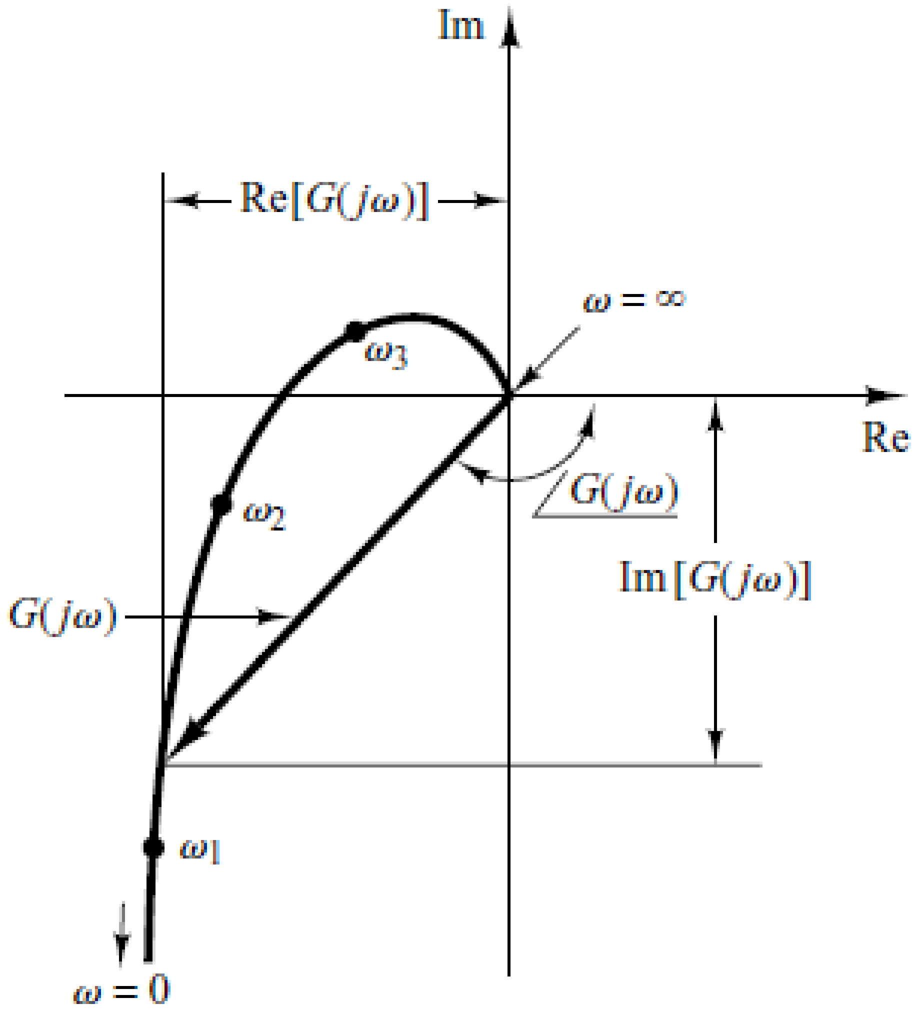
- The polar plot of a sinusoidal transfer function $G(j\omega)$ is a plot of the **magnitude of $G(j\omega)$** versus the **phase angle of $G(j\omega)$** on polar coordinates as ω is varied from zero to infinity.

- Thus, the polar plot is the **locus of vectors** $|G(j\omega)|\angle G(j\omega)$ as ω is varied from zero to infinity.

- Note that in polar plots, a **positive** (negative) phase angle is measured **counter-clockwise** (clockwise) from the positive real axis.

- The **polar plot** is often called the **Nyquist plot**.

Polar Plots (Nyquist)



Polar Plots (Nyquist)

Integrator: Draw the polar plot of the following transfer function

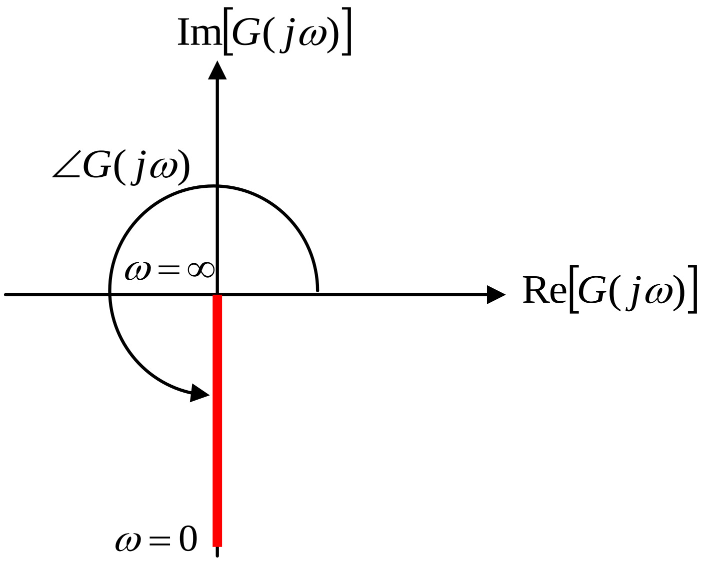
$$G(s) = \frac{1}{s}$$

$$G(j\omega) = \frac{1}{j\omega} \Rightarrow G(j\omega) = 0 - j\frac{1}{\omega}$$

$$\mathrm{Re}[G(j\omega)] = 0 \quad \& \quad \mathrm{Im}[G(j\omega)] = \frac{-1}{\omega}$$

$$\Rightarrow \begin{cases} \omega \to 0 & \mathrm{Im} \to -\infty \\ \omega \to \infty & \mathrm{Im} \to 0 \end{cases}$$

Phase is always -90



Polar Plots (Nyquist)

First order: Draw the polar plot of the following transfer function

$$G(s)=\frac{1}{s+1}$$

$$G(j\omega)=\frac{1}{j\omega+1} \Rightarrow G(j\omega)=\frac{1-j\omega}{1+\omega^2} \Rightarrow G(j\omega)=\frac{1}{1+\omega^2}-j\frac{\omega}{1+\omega^2}$$

$$\mathrm{Re}[G(j\omega)]=\frac{1}{1+\omega^2} \Rightarrow \begin{cases}\omega\to 0 & \mathrm{Re}\to 1\\ \omega\to\infty & \mathrm{Re}\to 0\end{cases}$$

$$\mathrm{Im}[G(j\omega)]=\frac{-\omega}{1+\omega^2} \Rightarrow \begin{cases}\omega\to 0 & \mathrm{Im}\to 0\\ \omega\to\infty & \mathrm{Im}\to 0\end{cases}$$

Note that for all ω, $\mathrm{Re}>0$ and $\mathrm{Im}<0$

Polar Plots (Nyquist)

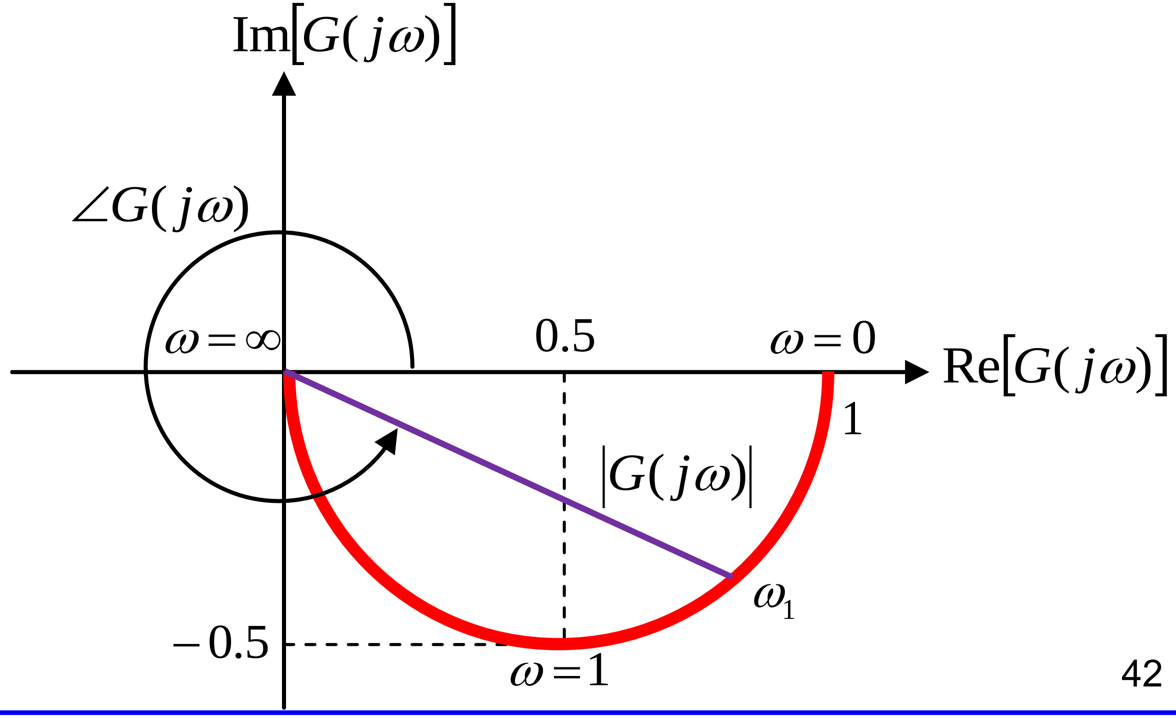
First order

$$G(s)=\frac{1}{s+1}$$

$$\mathrm{Re}[G(j\omega)]=\frac{1}{1+\omega^2}$$

$$\mathrm{Im}[G(j\omega)]=\frac{-\omega}{1+\omega^2}$$

$$\omega=1\Rightarrow\begin{cases}\mathrm{Re}\rightarrow 0.5\\ \mathrm{Im}\rightarrow -0.5\end{cases}$$



Polar Plots (Nyquist)

Second order: Draw the polar plot of the following transfer function

$$G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$G(j\omega)=\frac{1}{\left(1-\frac{\omega^2}{\omega_n^2}\right)+j2\zeta\frac{\omega}{\omega_n}}$$

$$G(j\omega)=\frac{\left(1-\frac{\omega^2}{\omega_n^2}\right)}{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+4\zeta^2\frac{\omega^2}{\omega_n^2}}-j\frac{2\zeta\frac{\omega}{\omega_n}}{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+4\zeta^2\frac{\omega^2}{\omega_n^2}}$$

Polar Plots (Nyquist)

Second order: $$G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$\mathrm{Re}\left[G(j\omega)\right]=\frac{\left(1-\frac{\omega^2}{\omega_n^2}\right)}{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+4\zeta^2\frac{\omega^2}{\omega_n^2}}$$

$$\mathrm{Re}\left[G(j\omega)\right]=\begin{cases}1 & \omega\to 0\\ 0 & \omega=\omega_n\\ 0 & \omega\to\infty\end{cases}$$

$$\mathrm{Im}\left[G(j\omega)\right]=-\frac{2\zeta\frac{\omega}{\omega_n}}{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+4\zeta^2\frac{\omega^2}{\omega_n^2}}$$

$$\mathrm{Im}\left[G(j\omega)\right]=\begin{cases}0 & \omega\to 0\\ \frac{1}{2\zeta} & \omega=\omega_n\\ 0 & \omega\to\infty\end{cases}$$

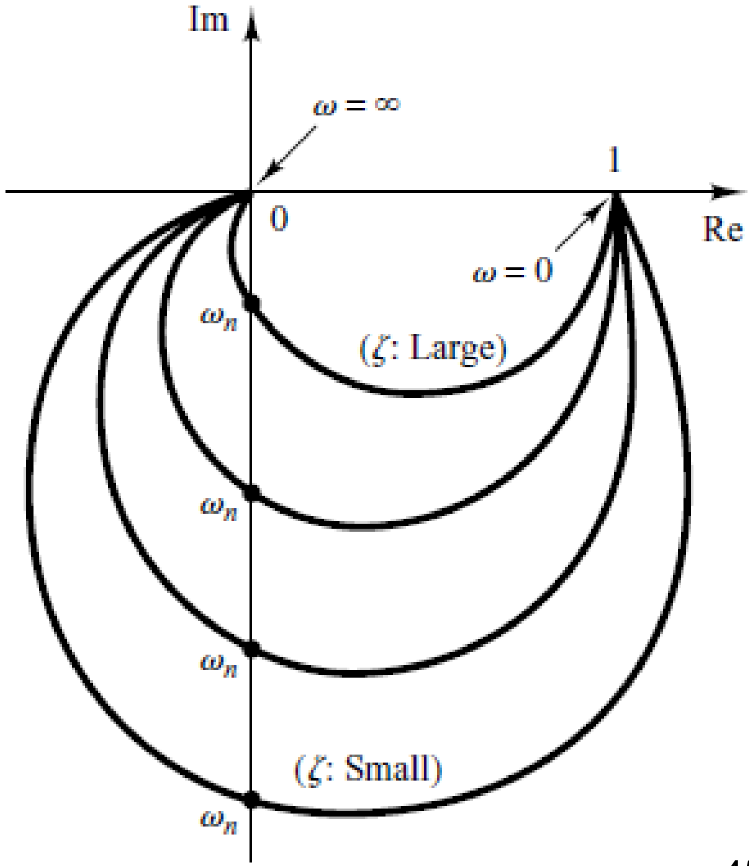
Polar Plots (Nyquist)

Second order:

$$G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$\mathrm{Re}\left[G(j\omega)\right]=\frac{\left(1-\dfrac{\omega^2}{\omega_n^2}\right)}{\left(1-\dfrac{\omega^2}{\omega_n^2}\right)^2+4\zeta^2\dfrac{\omega^2}{\omega_n^2}}$$

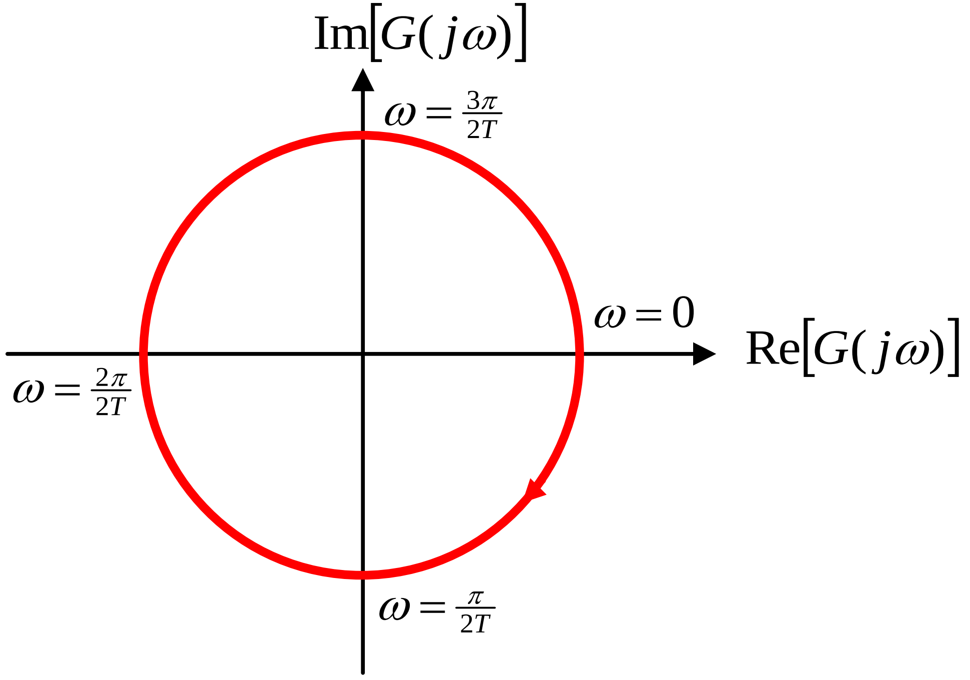
$$\mathrm{Im}\left[G(j\omega)\right]=-\frac{2\zeta\dfrac{\omega}{\omega_n}}{\left(1-\dfrac{\omega^2}{\omega_n^2}\right)^2+4\zeta^2\dfrac{\omega^2}{\omega_n^2}}$$



Polar Plots (Nyquist)

Transport lag: Draw the polar plot of the following transfer function

$G(s) = e^{-Ts}$ ⇒ $G(j\omega) = e^{-jT\omega}$ ⇒ $G(j\omega) = \cos(T\omega) - j\sin(T\omega)$



Polar Plots (Nyquist)

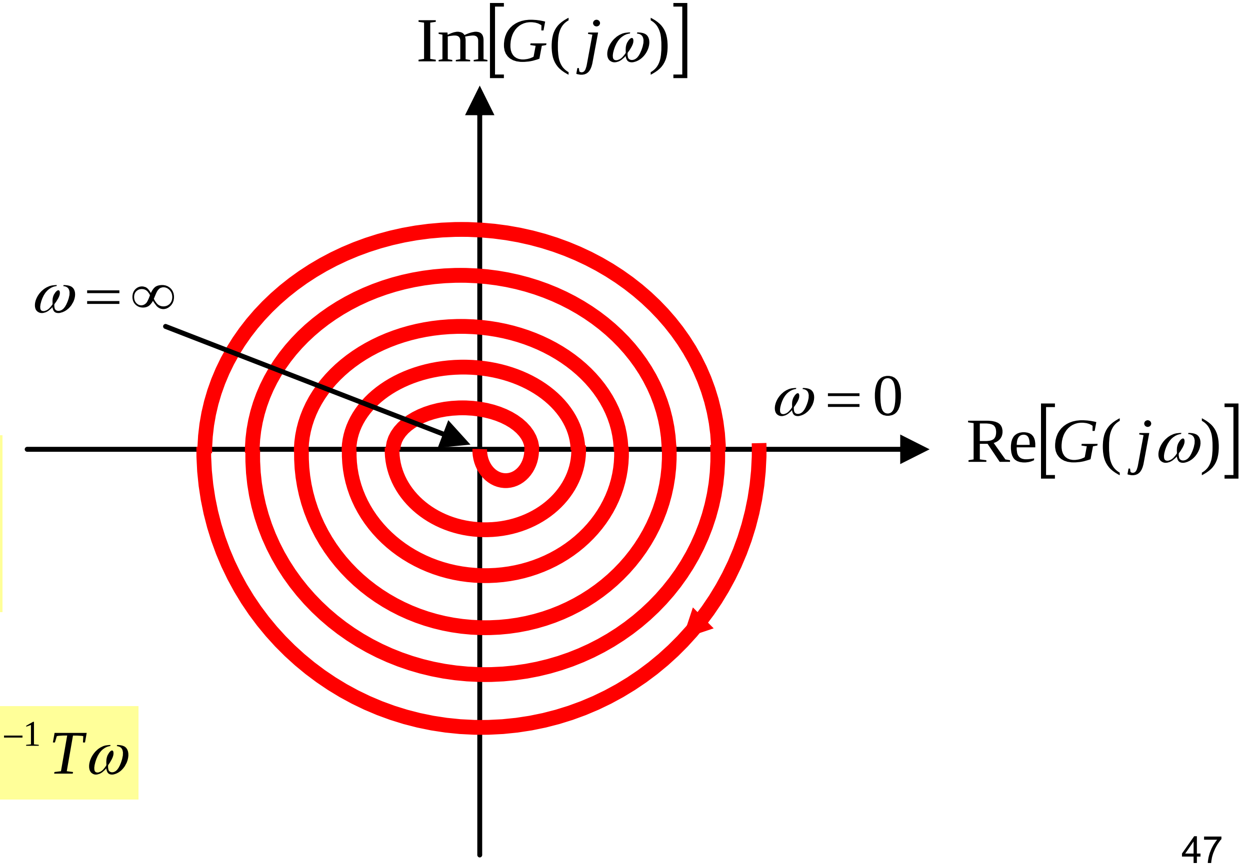
Example: Draw the polar plot of the following transfer function

$$G(s) = \frac{e^{-ks}}{1+Ts}$$

$$G(j\omega) = \frac{e^{-jk\omega}}{1+jT\omega}$$

$$|G(j\omega)| = \frac{1}{\sqrt{1+T^2\omega^2}}$$

$$\angle G(j\omega) = -k\omega - \tan^{-1} T\omega$$



Polar Plots (Nyquist)

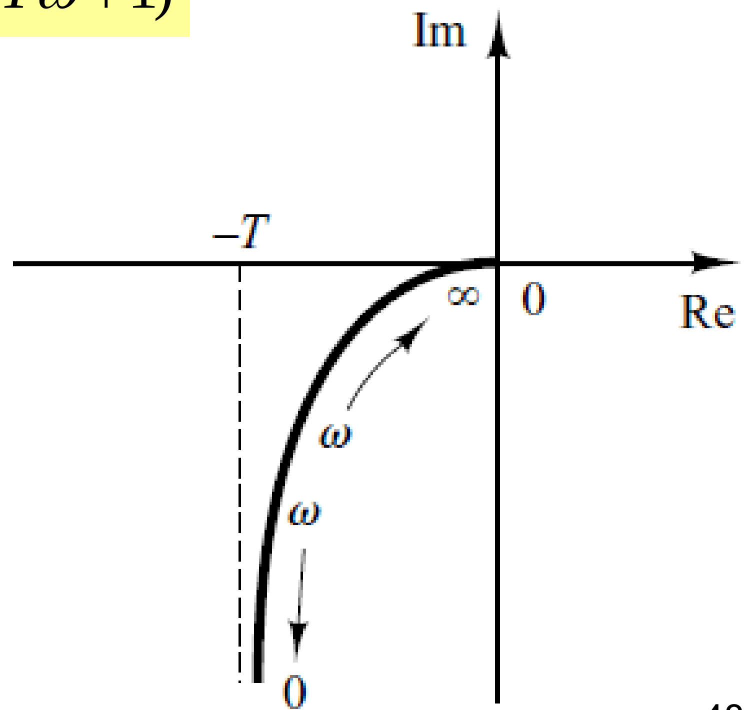
Example: Draw the polar plot of the following transfer function

$$G(s) = \frac{1}{s(Ts+1)} \Rightarrow G(j\omega) = \frac{1}{j\omega\,(jT\omega+1)}$$

$$\Rightarrow G(j\omega) = -\frac{T}{1+T^2\omega^2} - j\frac{1}{\omega(1+T^2\omega^2)}$$

$$\lim_{\omega\to 0} G(j\omega) = -T - j\infty$$

$$\lim_{\omega\to\infty} G(j\omega) = 0 - j0$$



General Shapes of Polar Plots

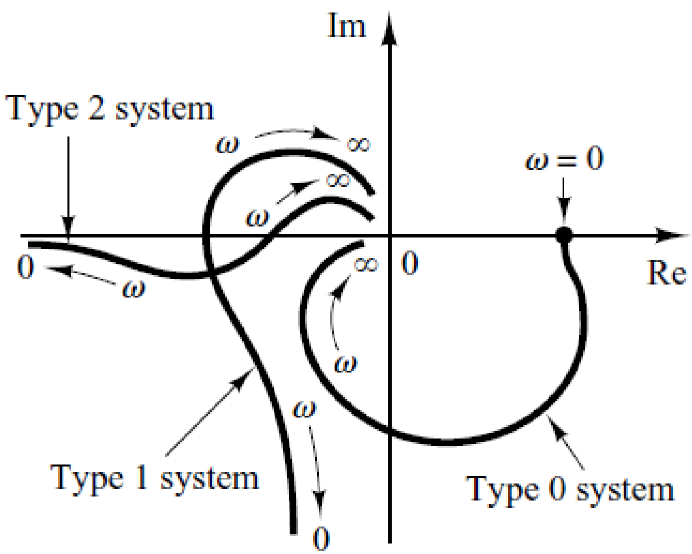
The polar plot of the following transfer function

$$G(j\omega)=\frac{(1+j\omega T_a)(1+j\omega T_b)\cdots}{(j\omega)^{\lambda}(1+j\omega T_1)(1+j\omega T_2)\cdots} \Rightarrow G(j\omega)=\frac{b_0(j\omega)^m+b_1(j\omega)^{m-1}+\cdots}{a_0(j\omega)^n+a_1(j\omega)^{n-1}+\cdots}$$

where $n>m$, will have the following general shapes:

1. ***For λ=0 or type 0 systems*:**

   The **starting point** of the polar plot (which corresponds to ω=0) is **finite** and is on the **positive real** axis.
   The **tangent** to the polar plot at ω=0 is **perpendicular** to the real axis.
   The **terminal point**, which corresponds to $\omega=\infty$, is at the **origin**, and the curve is tangent to one of the axes.

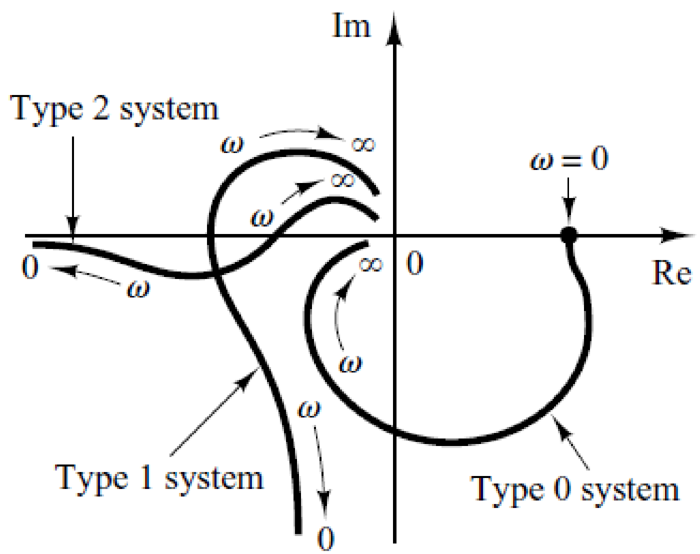


General Shapes of Polar Plots

2. *For λ=1 or type 1 systems:*
   At $\omega=0$, the magnitude of $G(j\omega)$ is **infinity**, and the **phase angle** becomes **–90°**.

   At **low frequencies**, the polar plot is **asymptotic** to a line parallel to the negative imaginary axis.

   At $\omega=\infty$, the magnitude becomes **zero**, and the curve converges to the **origin** and is tangent to one of the axes.

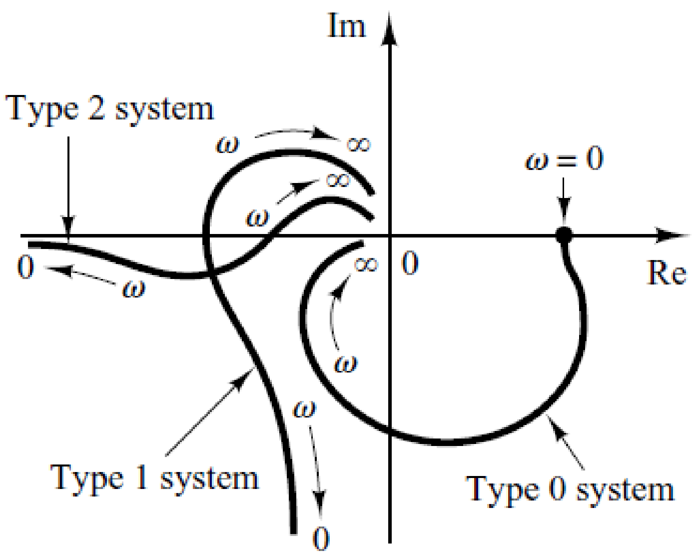


General Shapes of Polar Plots

2. *For λ=2 or type 2 systems:*
   At ω=0, the magnitude of $G(j\omega)$ is **infinity**, and the **phase angle** becomes **–180°**.

   At **low frequencies**, the polar plot may be **asymptotic** to the negative real axis.

   At $\omega = \infty$, the magnitude becomes **zero**, and the curve converges to the **origin** and is tangent to one of the axes.



Drawing Nyquist Plots with MATLAB

Consider a transfer function as

$$G(s) = \frac{\text{num}(s)}{\text{den}(s)}$$

The **Nyquist plot in MATLAB** is obtained using the following command:

nyquist(num,den,w)

Where **num** is the vector corresponding to the coefficients of the numerator, **den** is the vector corresponding to the coefficients of the denominator and **w** is the user-specified frequency vector.

Drawing Nyquist Plots with MATLAB

Example: Consider a transfer function as

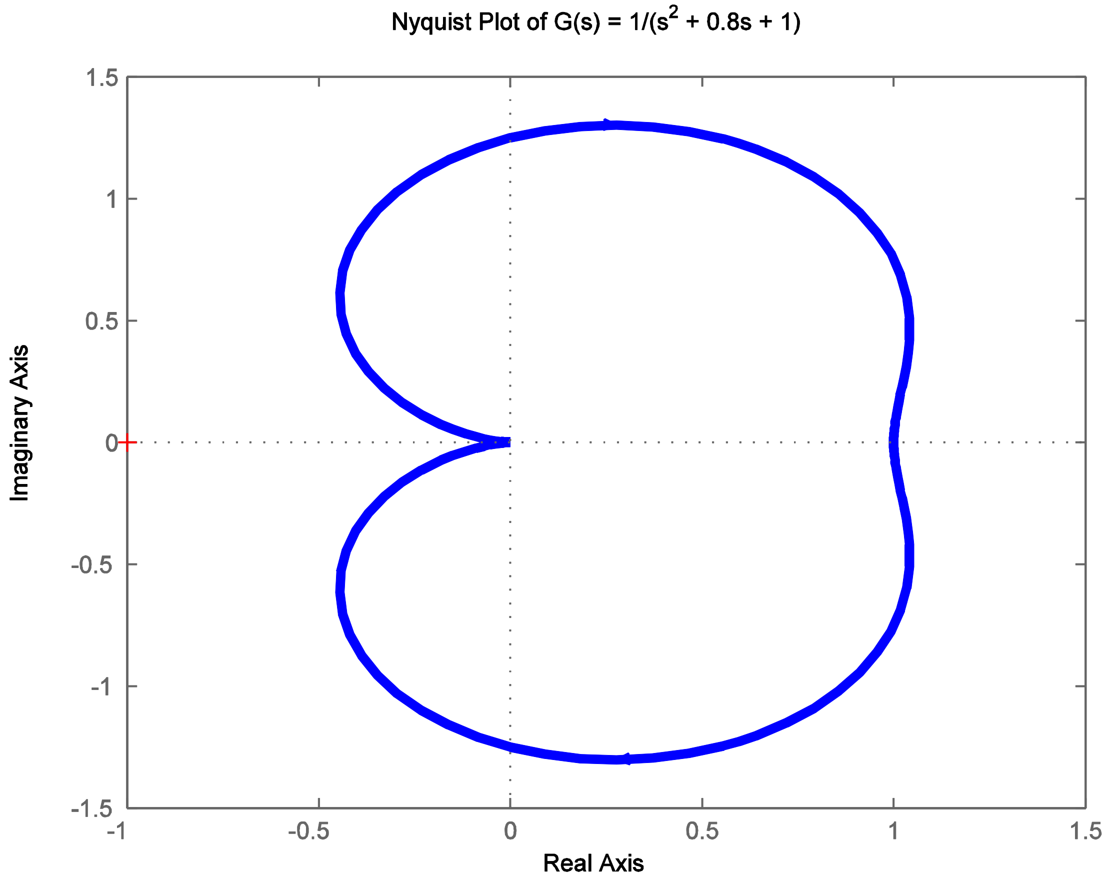
$$G(s) = \frac{1}{s^2 + 0.8s + 1}$$

The **Nyquist plot in MATLAB** is obtained using the following command:

```
num=[1];
den=[1 0.8 1];
nyquist(num,den)
title('Nyquist Plot of G(s) = 1/(s^2 + 0.8s + 1)')
```

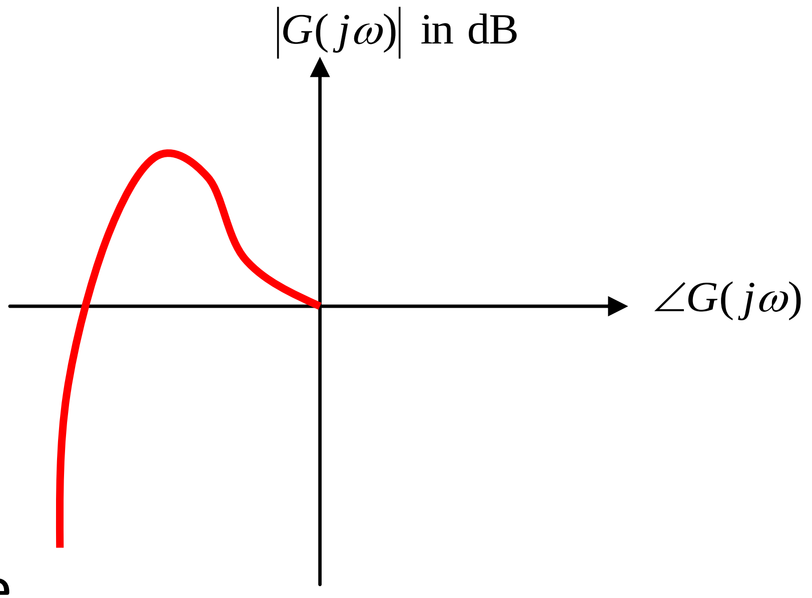
Drawing Nyquist Plots with MATLAB

Example: Consider a transfer function as $$G(s) = \frac{1}{s^2 + 0.8s + 1}$$



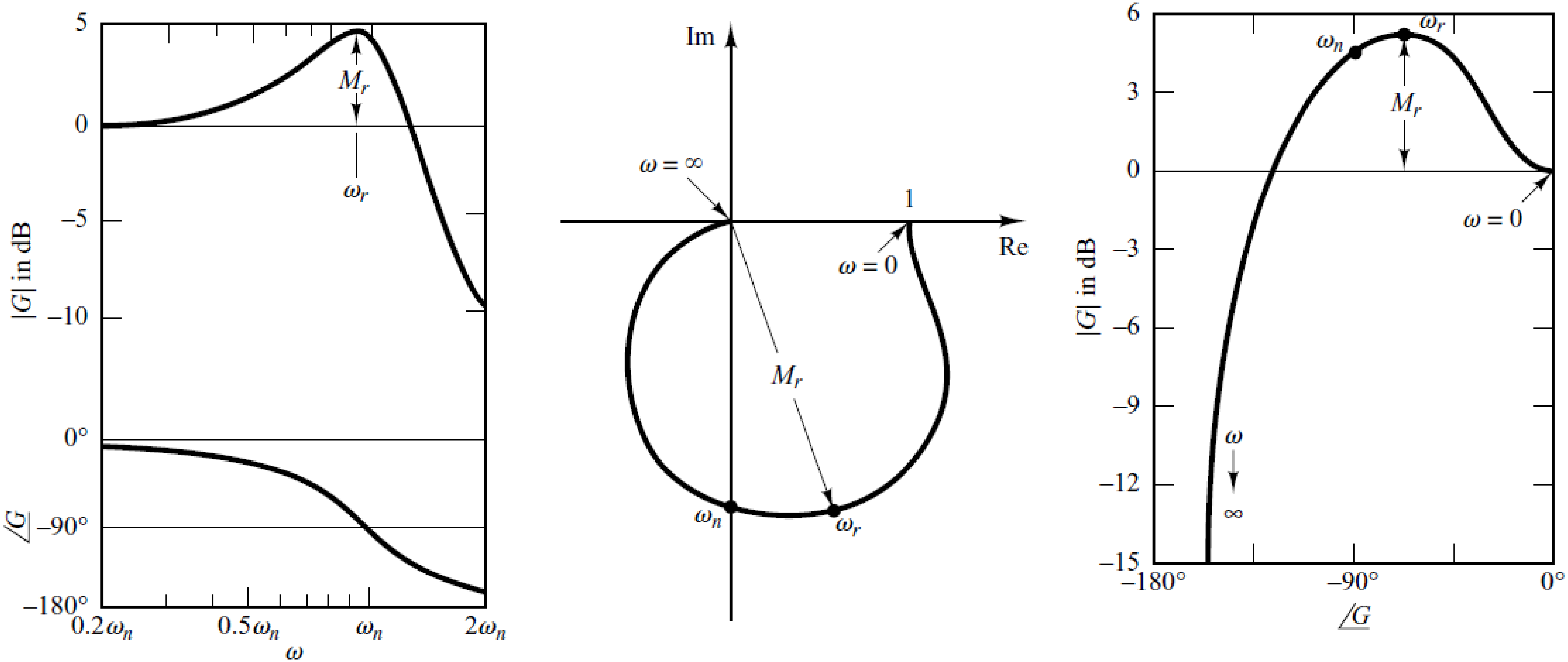
Log-Magnitude versus Phase Plots (Nichols Plots)

- Another approach to graphically portraying the frequency-response characteristics is to use the **log-magnitude-versus-phase plot**,
- which is a plot of the **logarithmic magnitude in decibels** versus the **phase angle**.
- In the log-magnitude-versus-phase plot, the **two curves** in the **Bode diagram** are **combined** into one.



Log-Magnitude versus Phase Plots (Nichols Plots)

(a) **Bode diagram**; (b) **polar plot**; (c) **log-magnitude-versus-phase plot** of a **second order system**.



(a) (b) (c)

Nyquist Stability Criterion

- The **Nyquist stability criterion** determines the **stability** of a **closed-loop system** from its **open-loop frequency response and open-loop poles**.
- Consider the following closed-loop transfer function

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

- For stability, all roots of the characteristic equation must lie in the left-half s plane. $1+G(s)H(s)=0$

- The **Nyquist stability criterion** **relates** the **open-loop frequency** response $G(j\omega)H(j\omega)$ to the **number of zeros and poles** of $1+G(s)H(s)$ that lie in the **right-half s plane**.

Conformal Mapping

- Consider the following open-loop transfer function

$$G(s)H(s) = \frac{2}{s-1}$$

- The characteristic equation is

$$F(s) = 1 + G(s)H(s) = 1 + \frac{2}{s-1} = \frac{s+1}{s-1} = 0$$

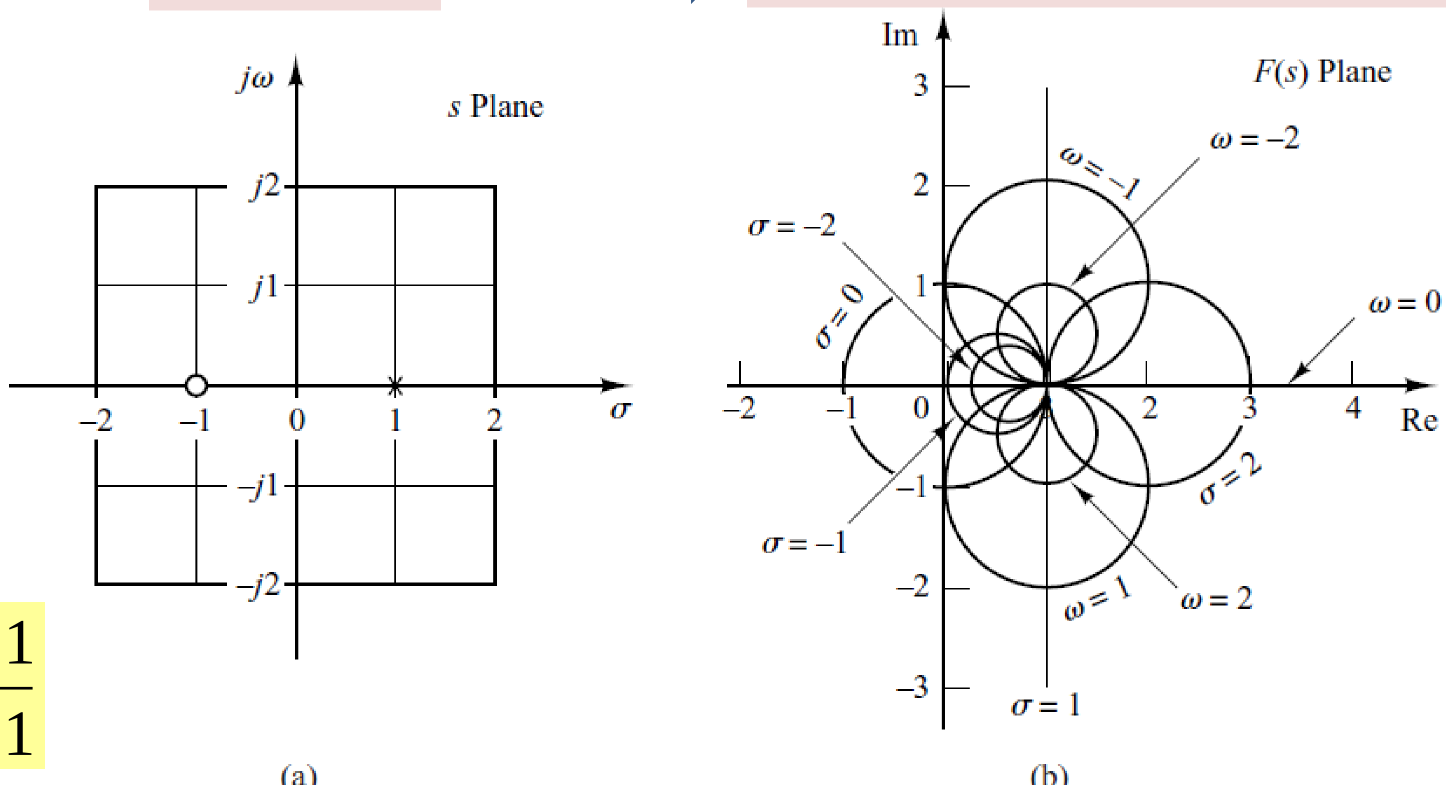
- The function $F(s)$ is analytic everywhere in the *s* plane except at its singular points.
- For each point of analyticity in the *s* plane, there corresponds a point in the $F(s)$ plane.
- For example, if $s=2+j1$, then $F(s)$ becomes

$$F(2+j1) = \frac{2+j1+1}{2+j1-1} = 2-j1$$

Conformal Mapping

For a given continuous closed path in the s plane, which does not go through any singular points, there corresponds a closed curve in the $F(s)$ plane.

$$s = \sigma + j\omega \xrightarrow{\text{Mapping}} F(s) = \mathrm{Re}[F(s)] + j\,\mathrm{Im}[F(s)]$$

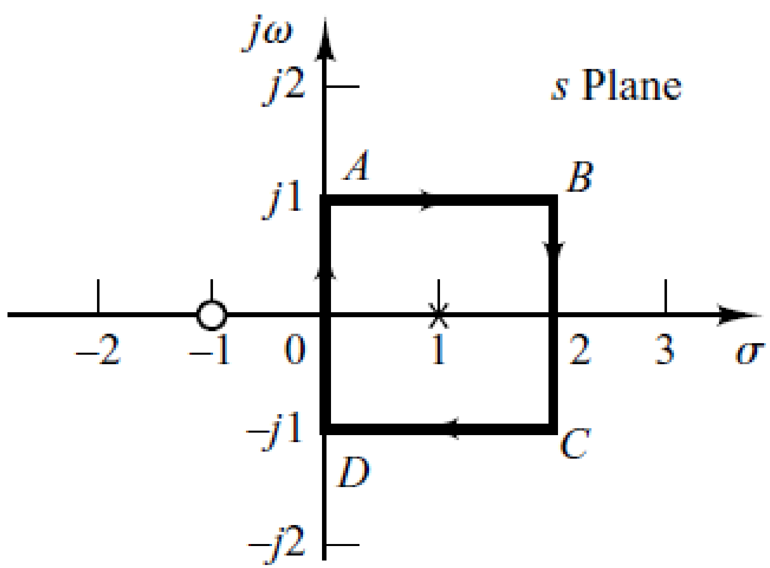


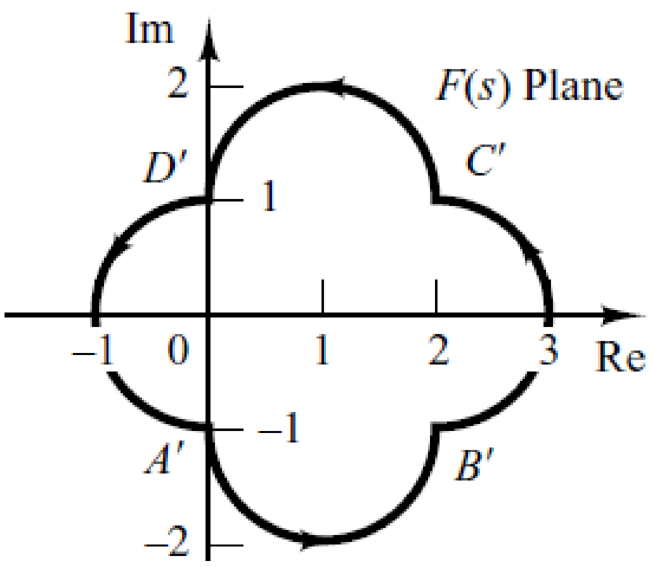
$$F(s) = \frac{s+1}{s-1}$$

Encirclement of the Origin

- Suppose that **representative point s traces out** a **contour** in the s plane in the **clockwise** direction.

1. If the contour in the s plane **encloses the pole** of $F(s)$, there is **one encirclement of the origin** of the $F(s)$ plane by the locus of $F(s)$ in the **counter-clockwise** direction.

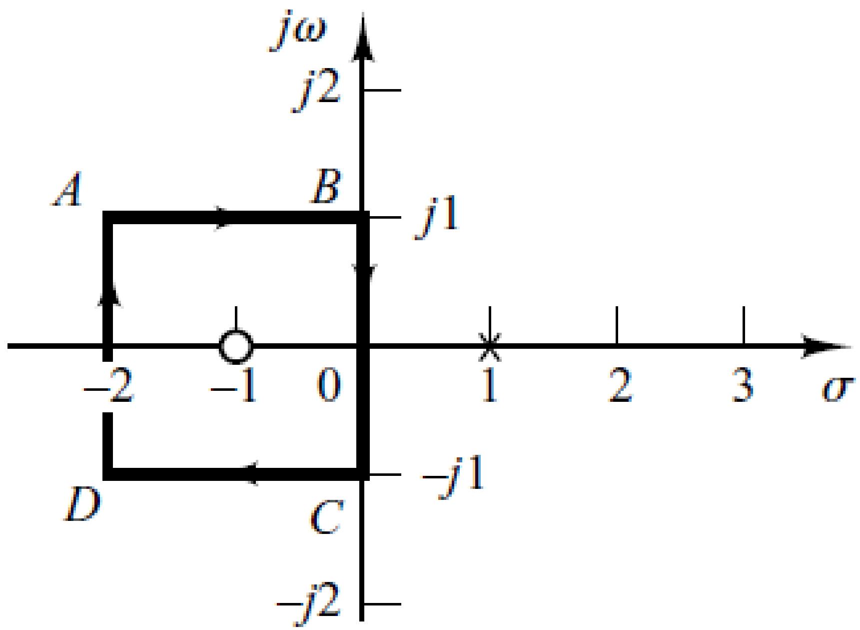


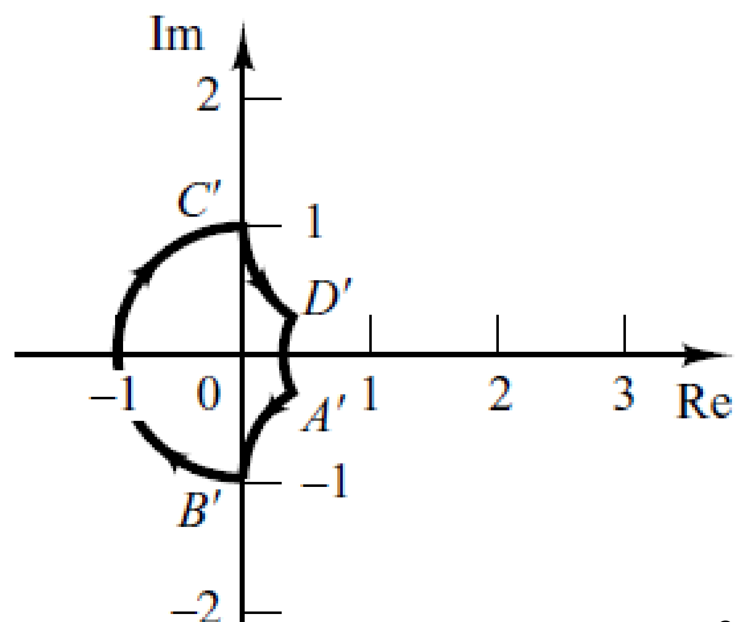


Encirclement of the Origin

- Suppose that **representative point s traces out** a **contour** in the s plane in the **clockwise** direction.

2. If the contour in the s plane **encloses the zero** of $F(s)$, there is **one encirclement of the origin** of the $F(s)$ plane by the locus of $F(s)$ in the **clockwise** direction.

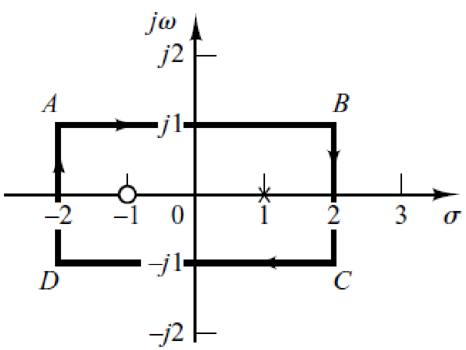


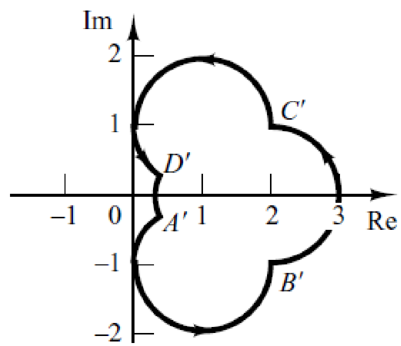


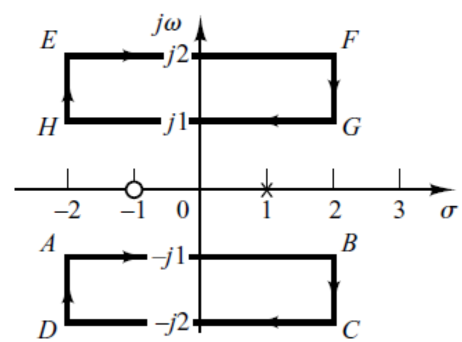
Encirclement of the Origin

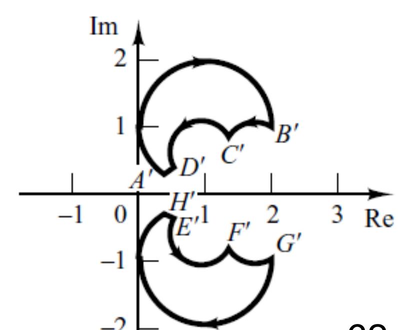
- Suppose that **representative point** s **traces out** a **contour** in the s plane in the **clockwise** direction.

3. If the contour in the s plane **encloses both the zero and the pole** **or** if the counter encloses **neither the zero nor the pole** of $F(s)$, then there is **no encirclement of the origin** of the $F(s)$ plane by the locus of $F(s)$.









Encirclement of the Origin

- The **direction of encirclement of the origin** of the $F(s)$ plane by the locus of $F(s)$ depends on whether the contour in the s plane **encloses a pole or a zero**.

- If the contour in the s plane **encloses equal numbers of poles and zeros**, then the corresponding closed curve in the $F(s)$ plane **does not encircle the origin** of the $F(s)$ plane.

Mapping

- Let $F(s)$ be a **ratio** of **two polynomials** in s.
- Let P be the **number of poles of $F(s)$** and Z be the **number of zeros of $F(s)$** that lie inside some closed contour in the s plane, with multiplicity of poles and zeros accounted for.
- Let the contour be such that it **does not pass** through any **poles or zeros** of $F(s)$.
- This closed contour in the s plane is then **mapped** into the $F(s)$ plane as a closed curve.
- The total number N of **clockwise encirclements of the origin** of the $F(s)$ plane, as a representative point s traces out the entire contour in the clockwise direction, is equal to Z-P.

$$N = Z - P$$

The mapping just gives the **difference of Z and P**, **NOT P and Z**

Mapping

$N = Z - P$

- $N > 0$ ⇒ $Z > P$ — Clockwise encirclements
- $N < 0$ ⇒ $Z < P$ — Counter-clockwise encirclements

- The number P can be readily determined for $F(s) = 1 + G(s)H(s)$ from the function $G(s)H(s)$.

- Therefore Z (**the number of poles of the closed-loop system lie inside some closed contour in the s plane**) can be found from P and N.

An Important Note

- Instead of mapping into $F(s) = 1 + G(s)H(s)$ the mapping is performed into $\Gamma(s) = G(s)H(s)$.

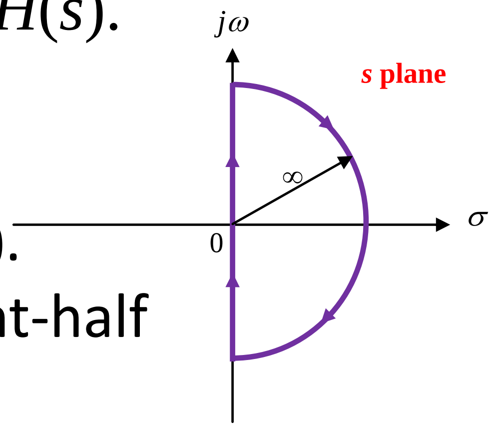
- Therefore, instead of counting the **number of clockwise encirclements of the origin**, the **number clockwise encirclements of the -1 point** is counted.

Procedure of Nyquist Stability Criterion

1. Form **loop transfer function** $G(s)H(s)$.

2. Form a semi-circle closed **contour** in the **right-half of s plane** that does not pass though the poles or zeros of $G(s)H(s)$.

   The direction of the semicircle is **clockwise**.



3. **Map** the contour in s plane into $\Gamma(s)=G(s)H(s)$.
4. Find the number of poles of $G(s)H(s)$ in the right-half s plane, i.e. P.
5. Count the number of clockwise encirclements of -1 point, i.e. N.
6. Find $Z = N + P$ which is the number of closed-loop poles in the right-half s plane.
7. If $Z=0$, the closed-loop system is **stable**.

Summary of Nyquist Stability Criterion

$$Z = N + P$$

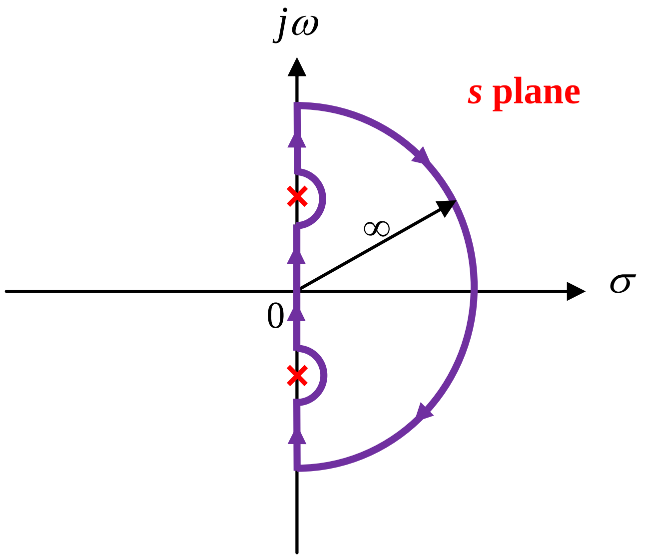
where

Z	number of zeros of $1+G(s)H(s)$ in the right-half s plane
N	number of clockwise encirclements of the $-1+j0$ point
P	number of poles of $G(s)H(s)$ in the right-half s plane

If $Z=0$, the closed-loop system is **stable**.

Some Points

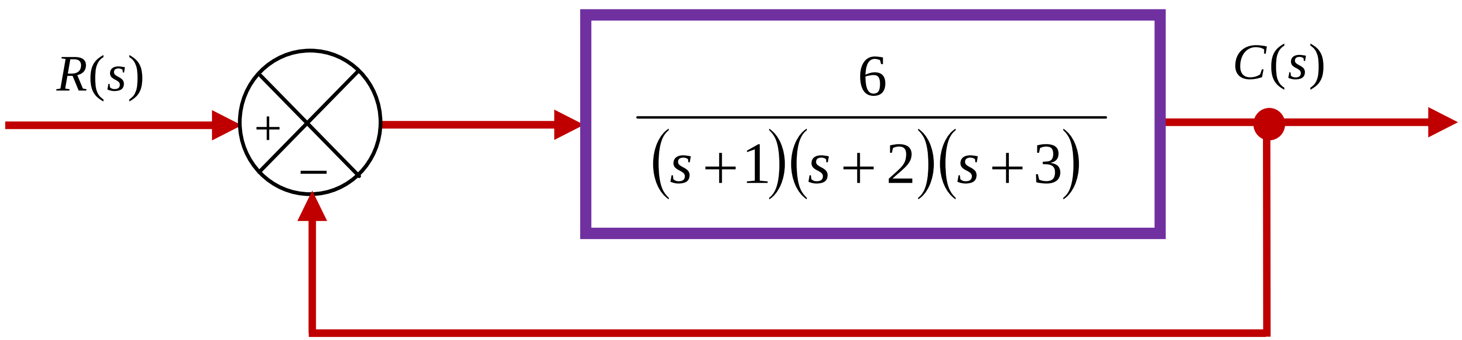
If there is any poles or zeros of $G(s)H(s)$ on the imaginary axis, the semi-circle in right-half of s plane should encircle them



If the locus of $G(j\omega)H(j\omega)$ passes through the $-1+j0$ point, then zeros of the characteristic equation, or closed-loop poles, are located on the $j\omega$ axis.

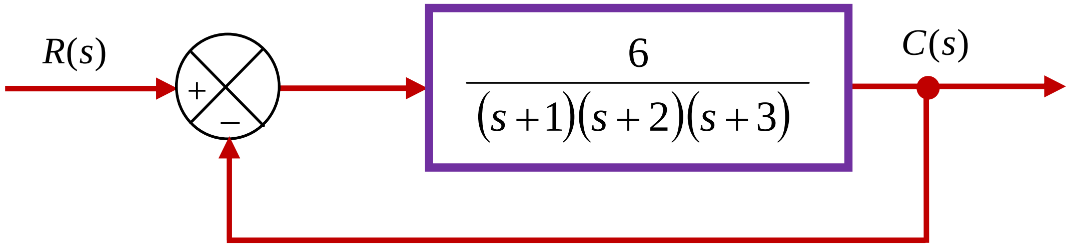
Nyquist Stability Criterion

- **Example**: Discuss on the stability of the following system using Nyquist stability criterion



Nyquist Stability Criterion

Solution:



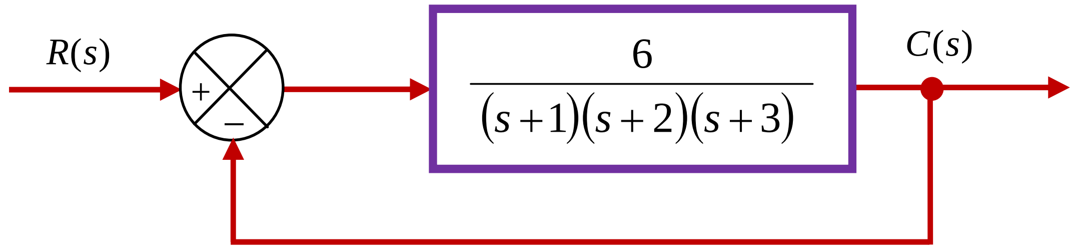
1. Form **loop transfer function** $G(s)H(s)$.

$$G(s)H(s) = \frac{6}{(s+1)(s+2)(s+3)}$$

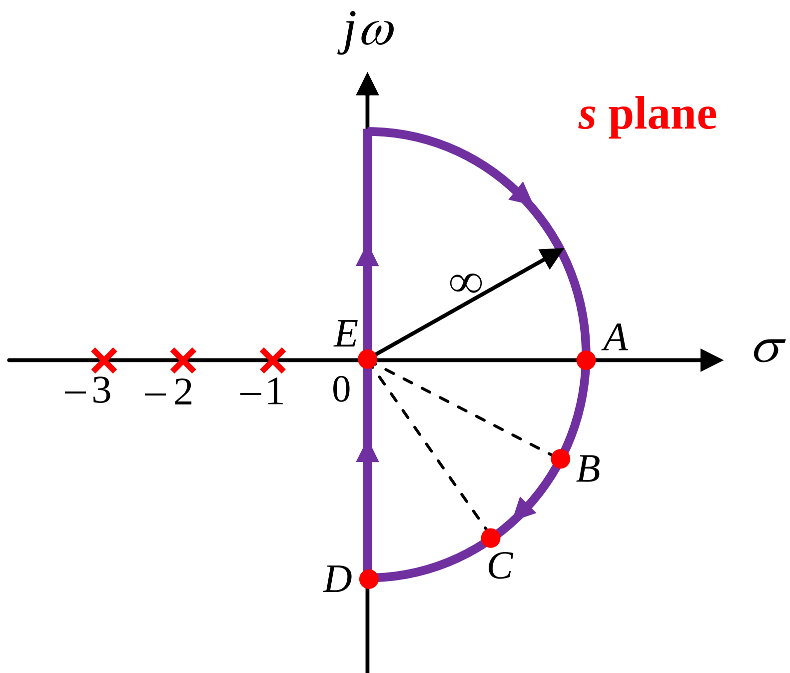
- The poles of $G(s)H(s)$ are $s=-1$ $s=-2$ $s=-3$
- $G(s)H(s)$ has no zero.

Nyquist Stability Criterion

Solution:



2. Form a semi-circle closed **contour** in the **right-half of s plane** that does not pass though the poles or zeros of $G(s)H(s)$.



Nyquist Stability Criterion

Solution:

3. **Map** the contour in s plane into $\Gamma(s)=G(s)H(s)$.

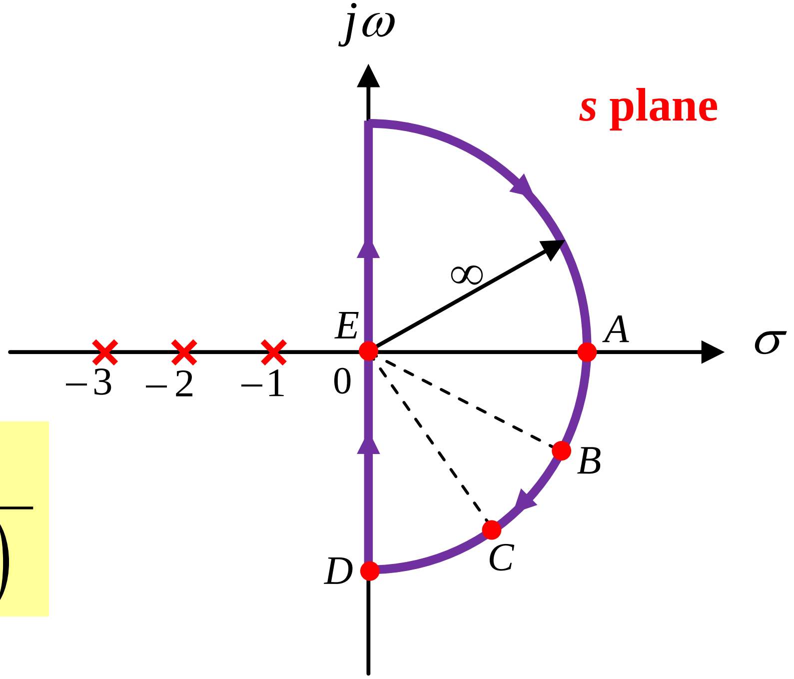
Section AD:

$$s = \lim_{R\to\infty} R\, e^{j\theta}$$

$$\Gamma(s) = G(s)H(s) = \frac{6}{(s+1)(s+2)(s+3)}$$

$$\Gamma\left(Re^{j\theta}\right) = \frac{6}{\left(Re^{j\theta}+1\right)\left(Re^{j\theta}+2\right)\left(Re^{j\theta}+3\right)}$$

$$\Gamma\left(Re^{j\theta}\right) = \frac{6}{R^3 e^{j3\theta}} = \varepsilon\, e^{-j3\theta}$$



Nyquist Stability Criterion

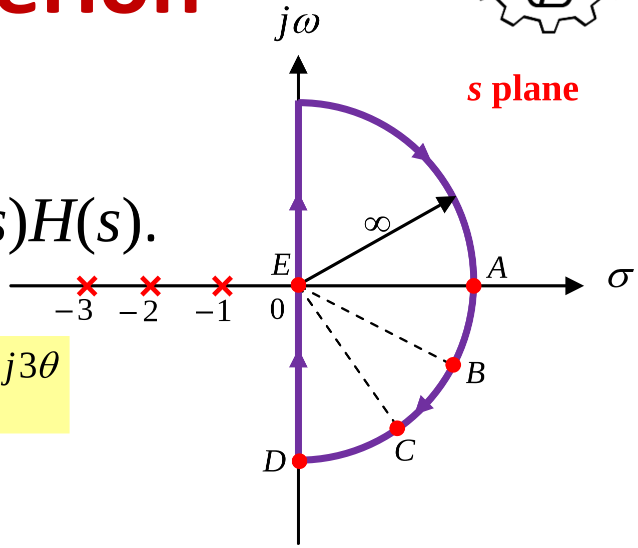
Solution:

3. **Map** the contour in s plane into $\Gamma(s)=G(s)H(s)$.

Section ***AD***:

$$s = \underset{R\to\infty}{R}\, e^{j\theta}$$

$$\Gamma\left(R\,e^{j\theta}\right) = \varepsilon\, e^{-j3\theta}$$

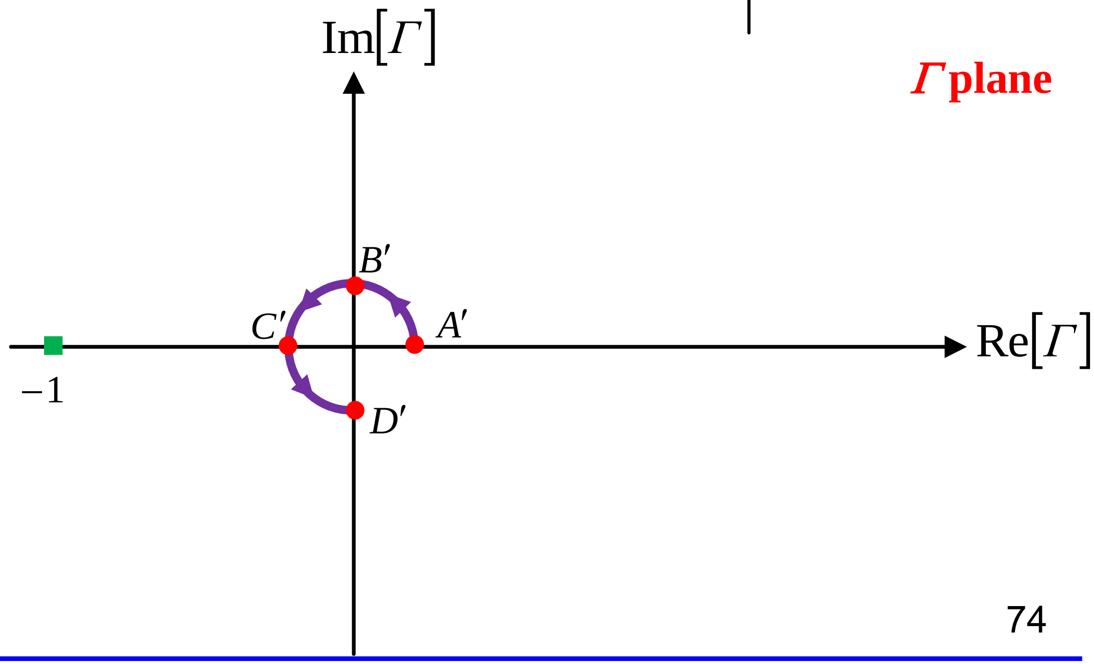


$A \to A' \qquad \Gamma = \varepsilon\, e^{j0}$

$B \to B' \qquad \Gamma = \varepsilon\, e^{j\pi/2}$

$C \to C' \qquad \Gamma = \varepsilon\, e^{j\pi}$

$D \to D' \qquad \Gamma = \varepsilon\, e^{j3\pi/2}$



Nyquist Stability Criterion

Solution:

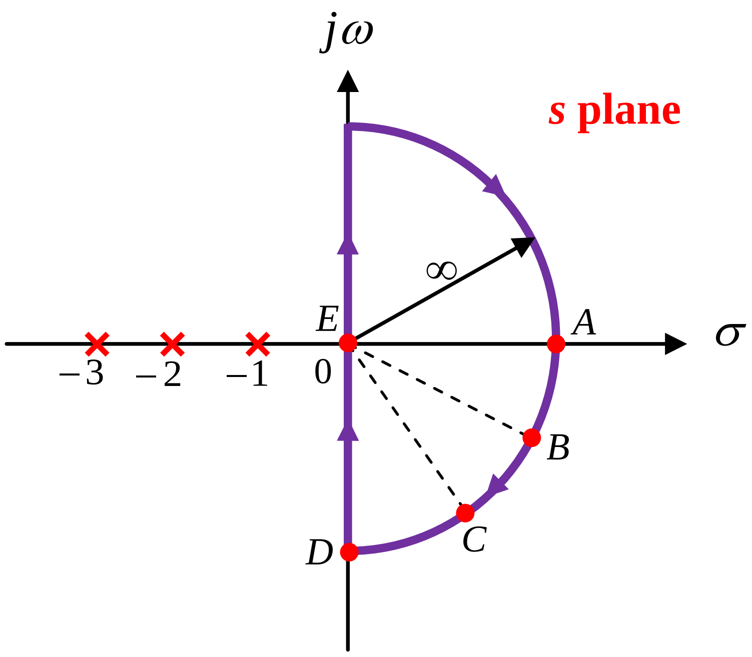
3. **Map** the contour in s plane into $\Gamma(s)=G(s)H(s)$.

Section *DE*: $s=-j\omega$

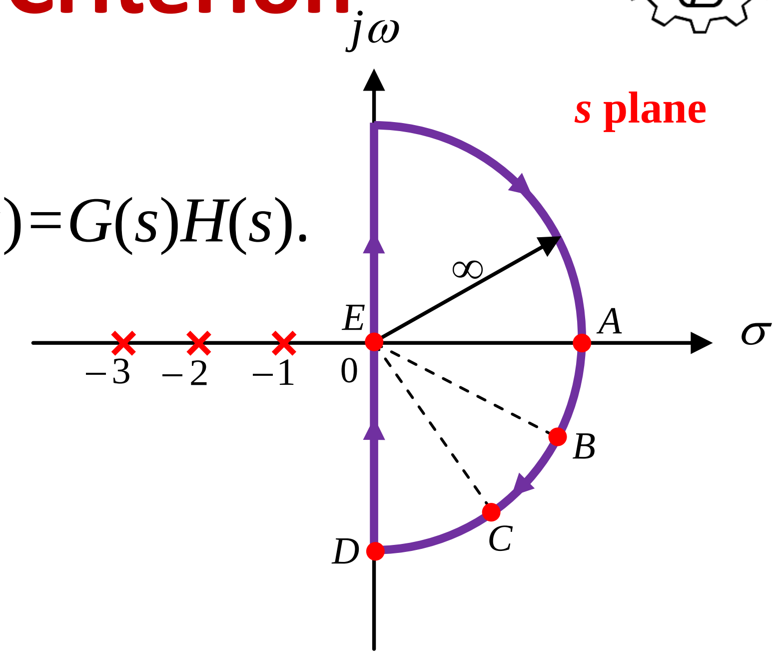
$$\Gamma(s)=G(s)H(s)=\frac{6}{(s+1)(s+2)(s+3)}$$

$$\Gamma(j\omega)=\frac{6}{(-j\omega+1)(-j\omega+2)(-j\omega+3)}$$

$$\Gamma(j\omega)=\frac{6}{6(1-\omega^2)-j\omega(11-\omega^2)}$$



Nyquist Stability Criterion



Solution:

3. **Map** the contour in *s* plane into $\Gamma(s)=G(s)H(s)$.

Section *DE*: $s=-j\omega$

$$\Gamma(j\omega)=\frac{6}{6\left(1-\omega^2\right)-j\omega\left(11-\omega^2\right)}$$

$$\mathrm{Re}\left[\Gamma(j\omega)\right]=\frac{36\left(1-\omega^2\right)}{36\left(1-\omega^2\right)^2+\omega^2\left(11-\omega^2\right)^2}$$

$$\mathrm{Im}\left[\Gamma(j\omega)\right]=\frac{6\omega\left(11-\omega^2\right)}{36\left(1-\omega^2\right)^2+\omega^2\left(11-\omega^2\right)^2}$$

Nyquist Stability Criterion

Solution:

3. **Map** the contour in s plane into $\Gamma(s)=G(s)H(s)$.

Section DE: $s=-j\omega$

$$\mathrm{Re}[\Gamma(j\omega)]=\frac{36(1-\omega^2)}{36(1-\omega^2)^2+\omega^2(11-\omega^2)^2}=\begin{cases}0 & \omega\to\infty\\ -0.1 & \omega=\sqrt{11}\\ 0 & \omega=1\\ 1 & \omega=0\end{cases}$$

$$\mathrm{Im}[\Gamma(j\omega)]=\frac{6\omega(11-\omega^2)}{36(1-\omega^2)^2+\omega^2(11-\omega^2)^2}=\begin{cases}0 & \omega\to\infty\\ 0 & \omega=\sqrt{11}\\ 0.6 & \omega=1\\ 0 & \omega=0\end{cases}$$

Nyquist Stability Criterion

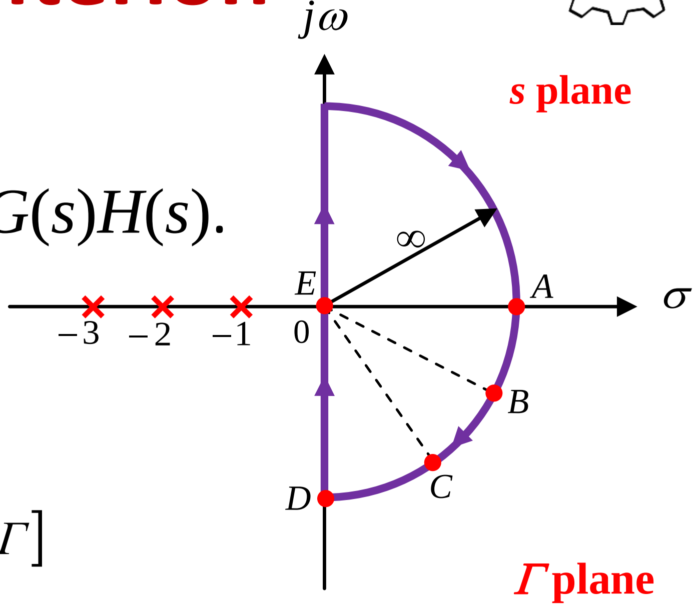
Solution:

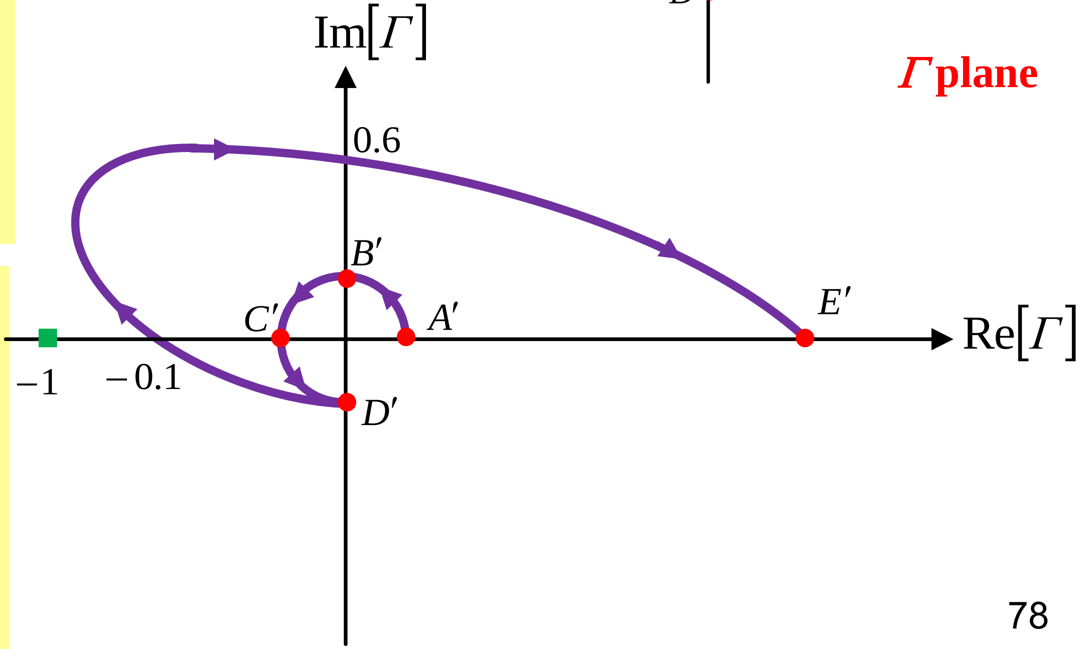
3. **Map** the contour in s plane into $\Gamma(s)=G(s)H(s)$.

Section DE: $s=-j\omega$

$$\mathrm{Re}\left[\Gamma(j\omega)\right]=\begin{cases}0 & \omega\to\infty\\ -0.1 & \omega=\sqrt{11}\\ 0 & \omega=1\\ 1 & \omega=0\end{cases}$$

$$\mathrm{Im}\left[\Gamma(j\omega)\right]=\begin{cases}0 & \omega\to\infty\\ 0 & \omega=\sqrt{11}\\ 0.6 & \omega=1\\ 0 & \omega=0\end{cases}$$

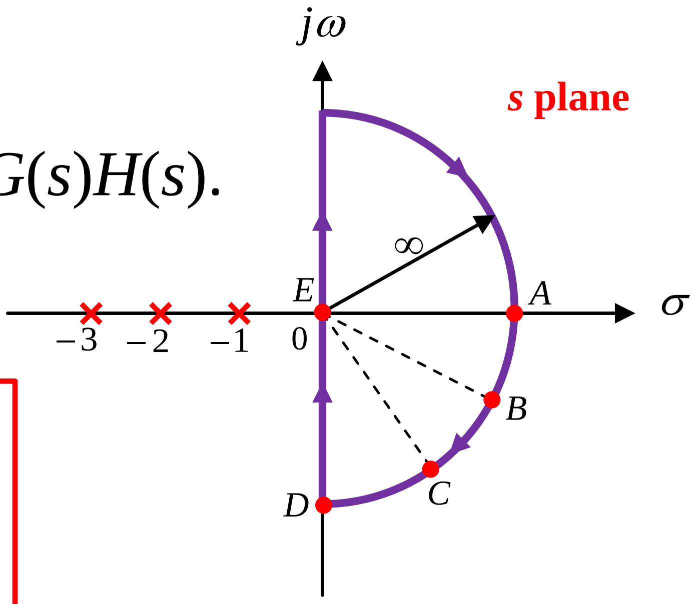


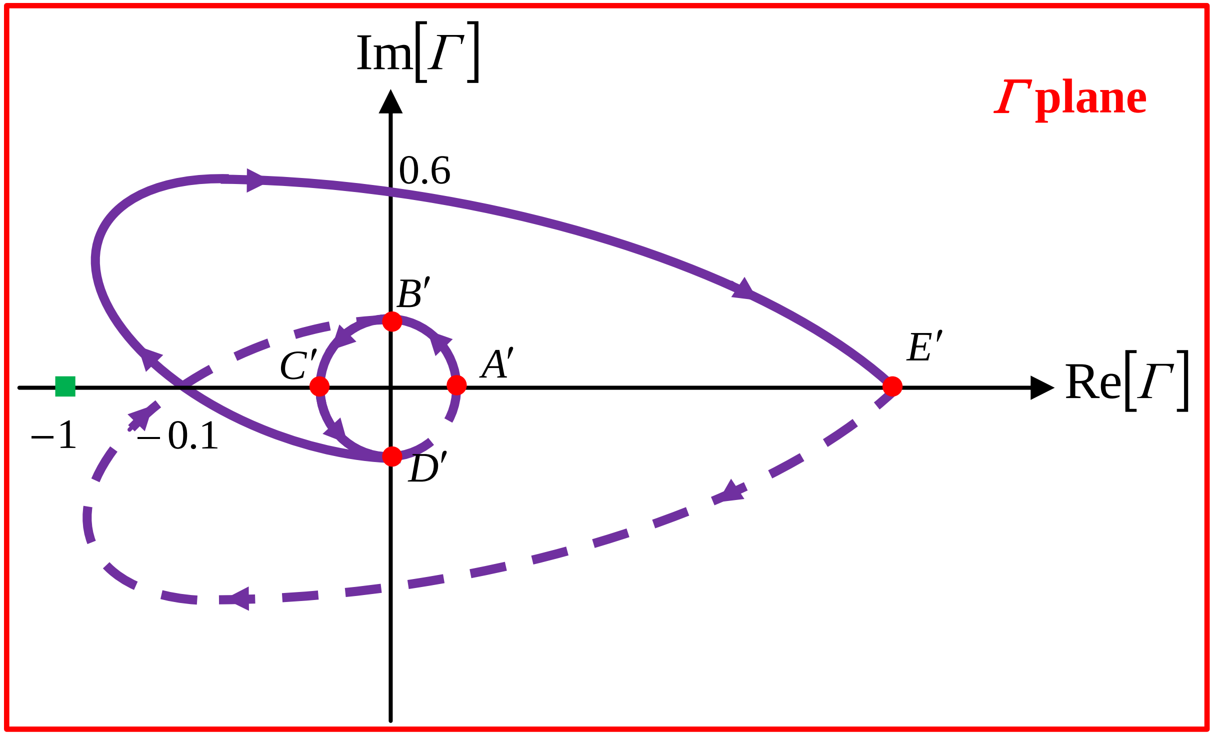


Nyquist Stability Criterion

Solution:

3. **Map** the contour in s plane into $\Gamma(s) = G(s)H(s)$.

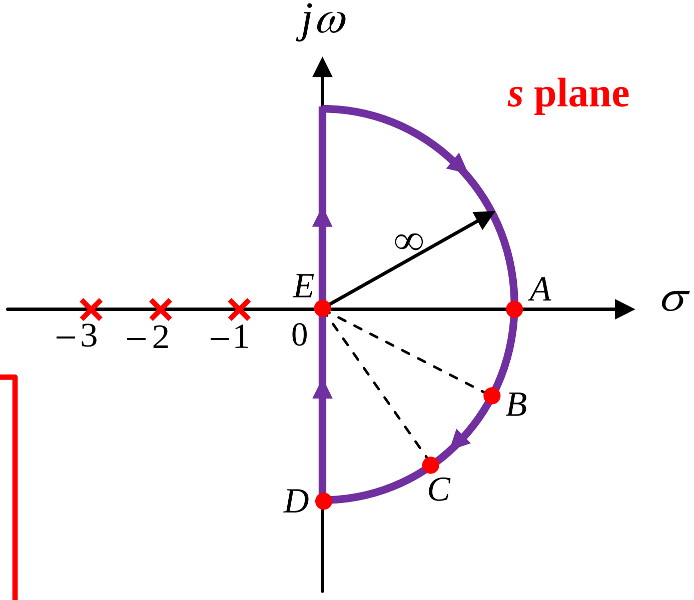




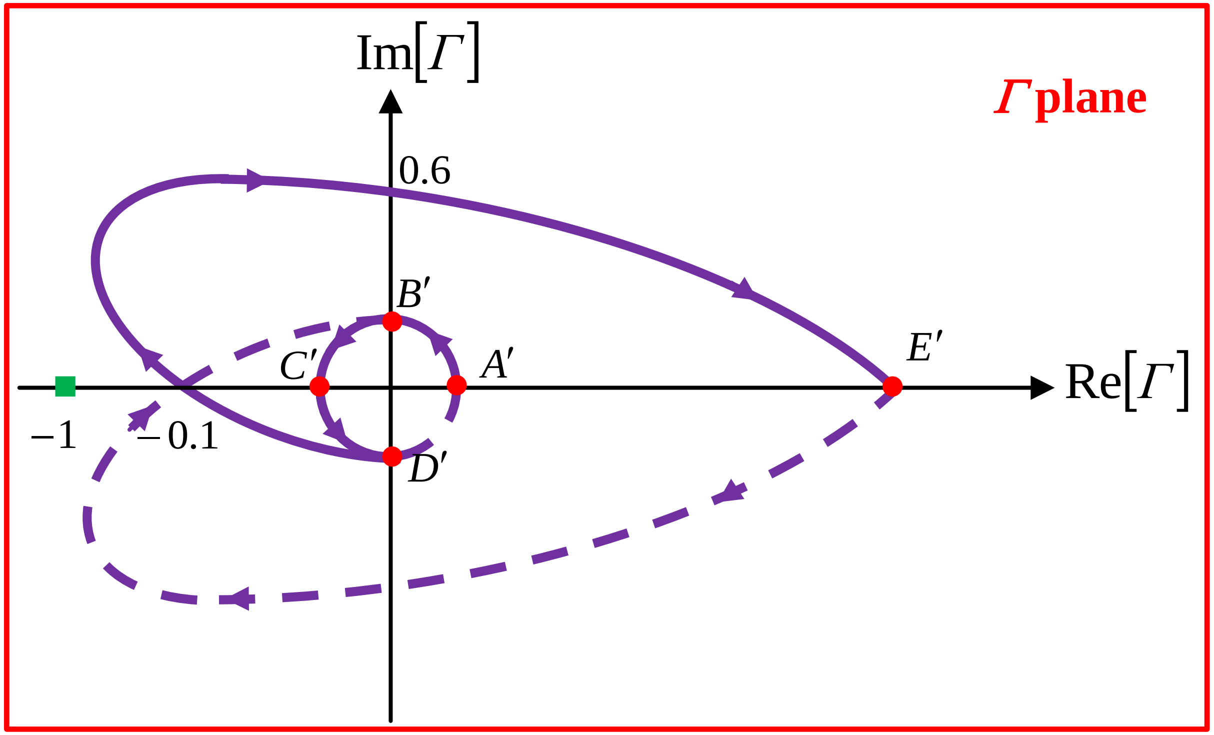
Nyquist Stability Criterion

Solution:

4. Find the number of poles of $G(s)H(s)$ in the right-half s plane, i.e. $\boldsymbol{P}$.



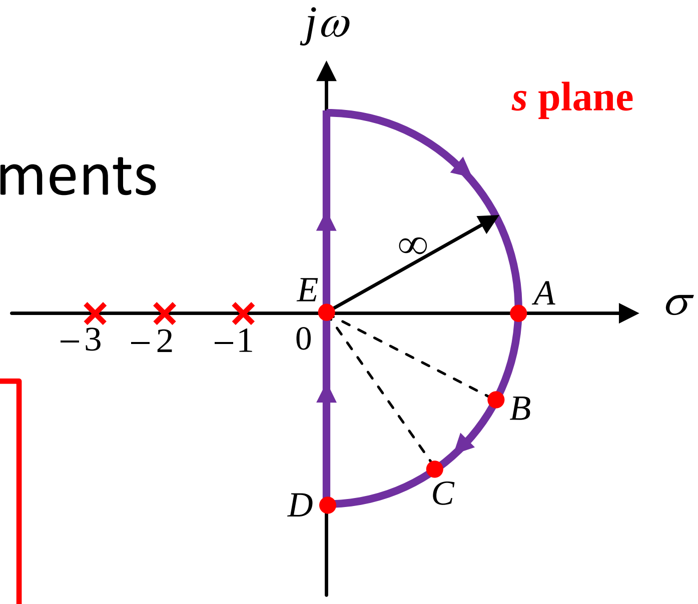
$P = 0$

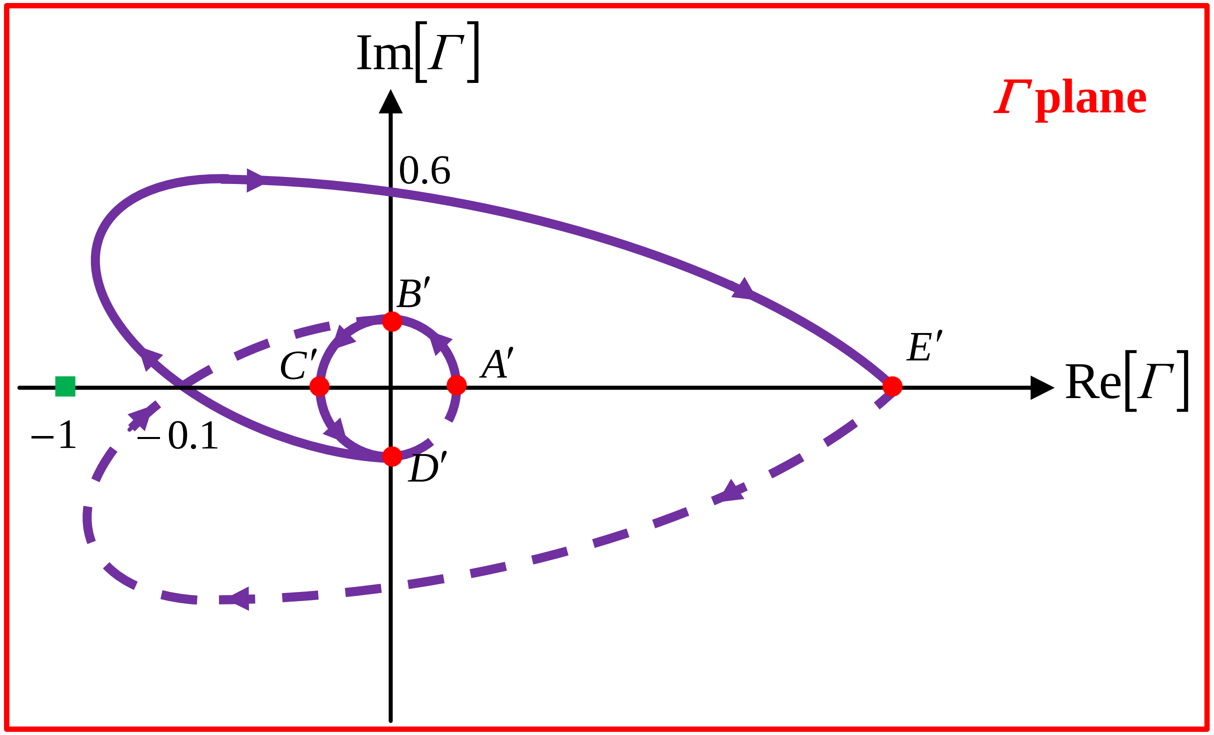


Nyquist Stability Criterion

Solution:

5. Count the number of clockwise encirclements of -1 point, i.e. ***N***.



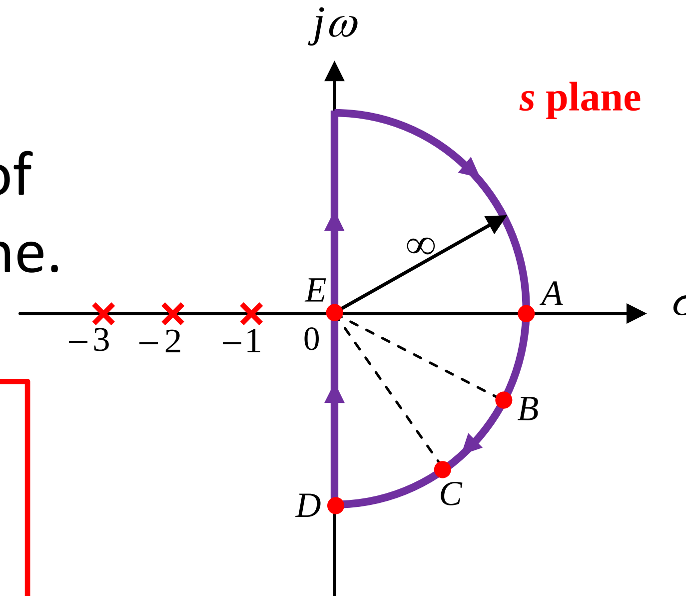


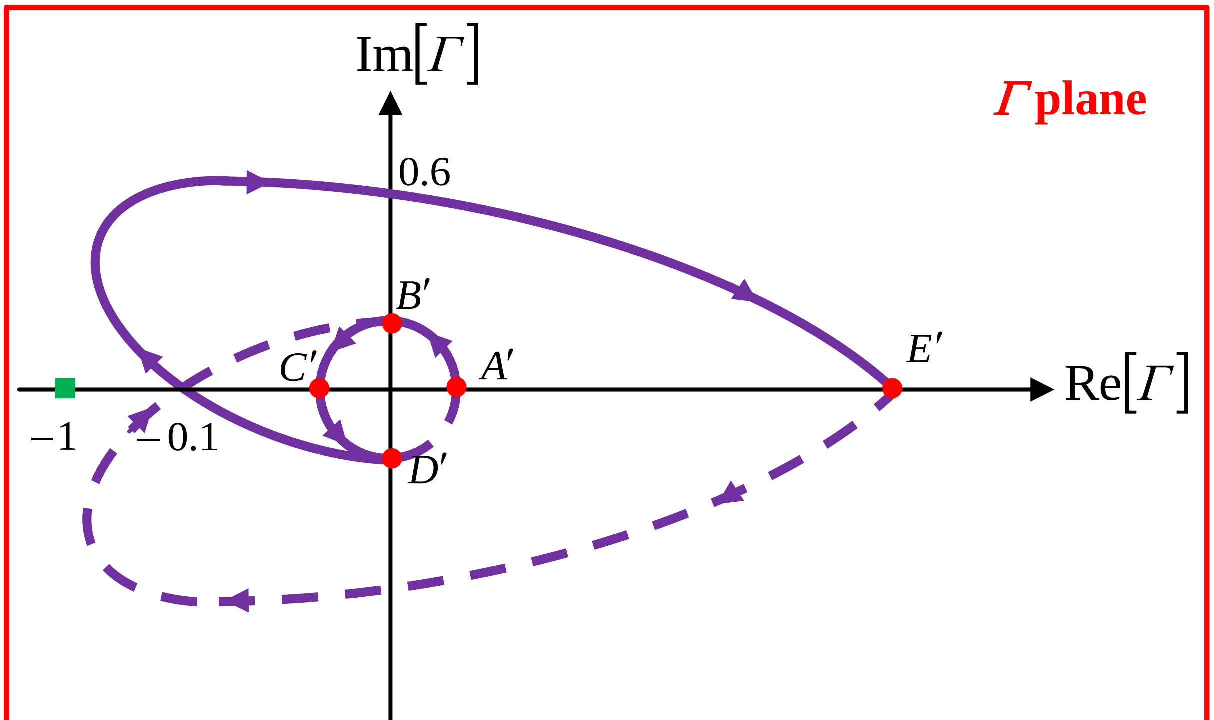
$N = 0$

Nyquist Stability Criterion

Solution:

6. Find $Z = N + P$ which is the number of closed-loop poles in the right-half s plane.





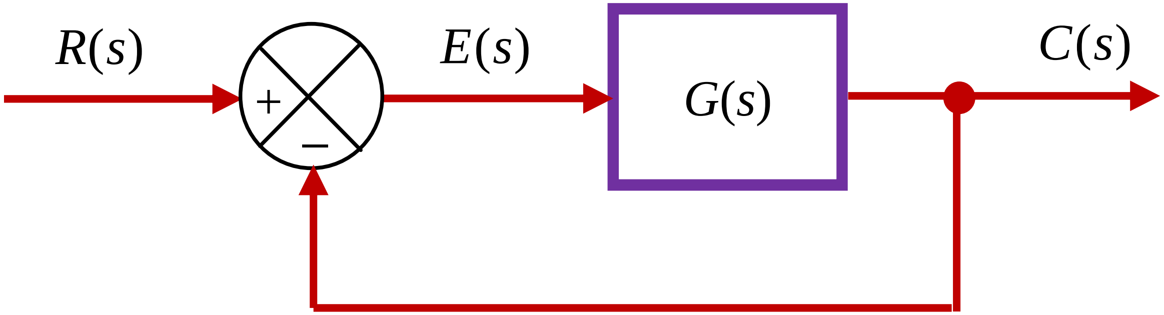
$Z = 0$

The system is stable.

Nyquist Stability Criterion

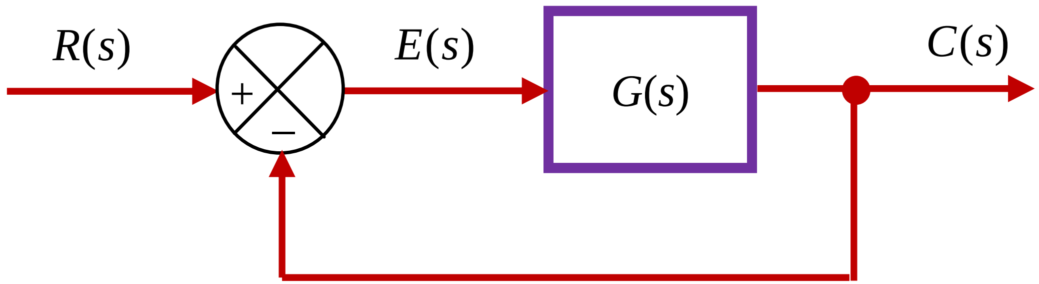
- **Example**: Discuss on the stability of the unity feedback system with the following forward path transfer function using Nyquist stability criterion

$$G(s) = \frac{s-1}{s(s+1)}$$



Nyquist Stability Criterion

Solution:



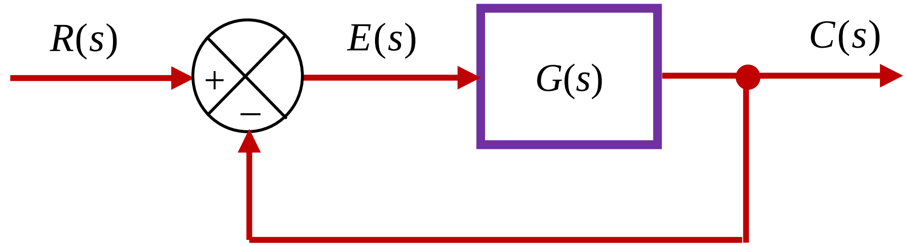
1. Form **loop transfer function** $G(s)H(s)$.

$$G(s)H(s)=\frac{s-1}{s(s+1)}$$

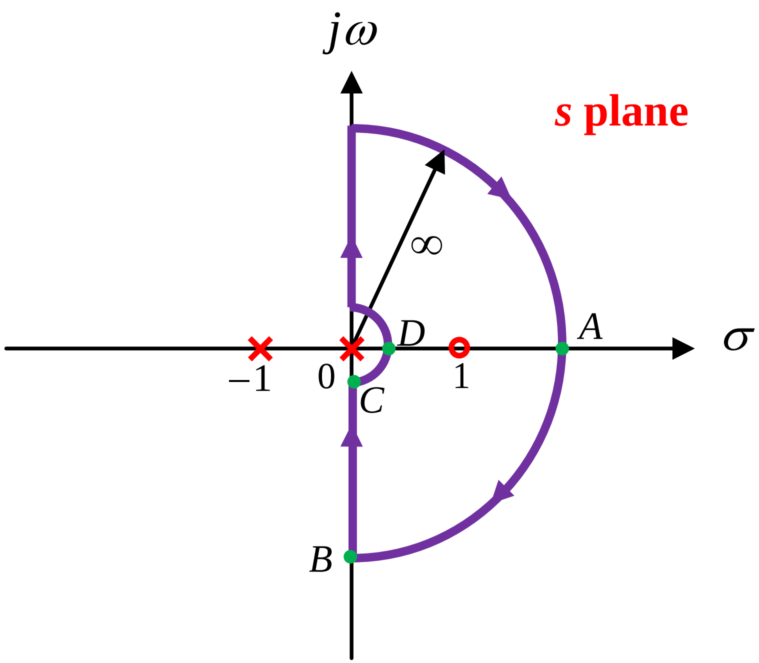
- The poles of $G(s)H(s)$ are $s=0$ $s=-1$
- The zero of $G(s)H(s)$ is $s=1$

Nyquist Stability Criterion

Solution:



2. Form a semi-circle closed **contour** in the **right-half of s plane** that does not pass though the poles or zeros of $G(s)H(s)$.



Nyquist Stability Criterion

Solution:

3. **Map** the contour in s plane into $\Gamma(s)=G(s)H(s)$.

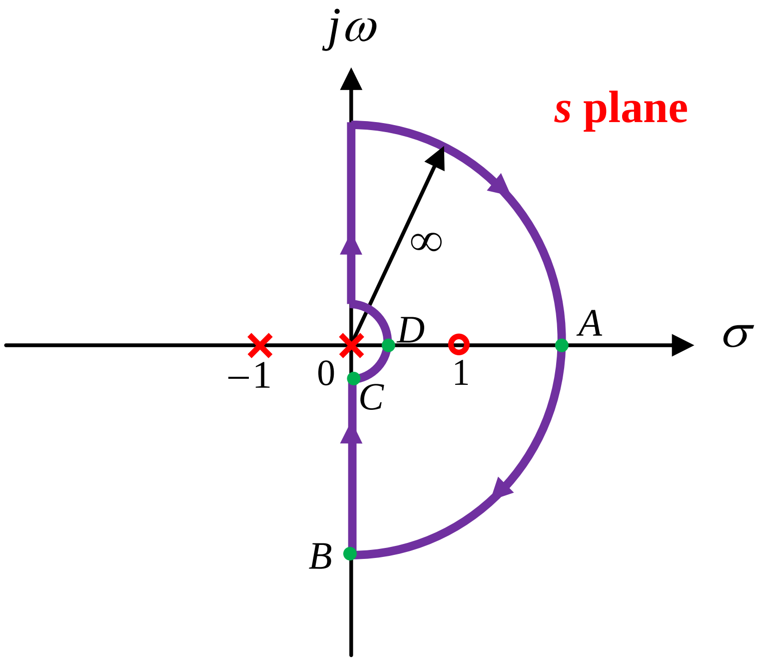
Section AB:

$$s = \lim_{R\to\infty} R\, e^{j\theta}$$

$$\Gamma(s) = G(s)H(s) = \frac{(s-1)}{s(s+1)}$$

$$\Gamma\left(Re^{j\theta}\right) = \frac{\left(Re^{j\theta}-1\right)}{Re^{j\theta}\left(Re^{j\theta}+1\right)}$$

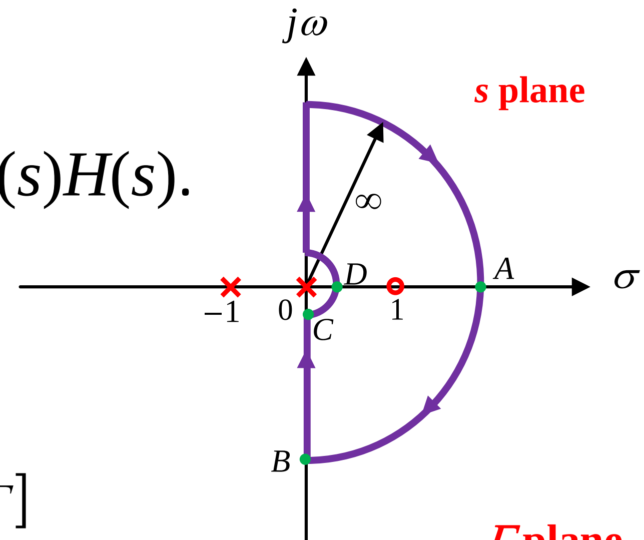
$$\Gamma\left(Re^{j\theta}\right) = \varepsilon\, e^{-j\theta}$$



Nyquist Stability Criterion

Solution:

3. **Map** the contour in s plane into $\Gamma(s) = G(s)H(s)$.



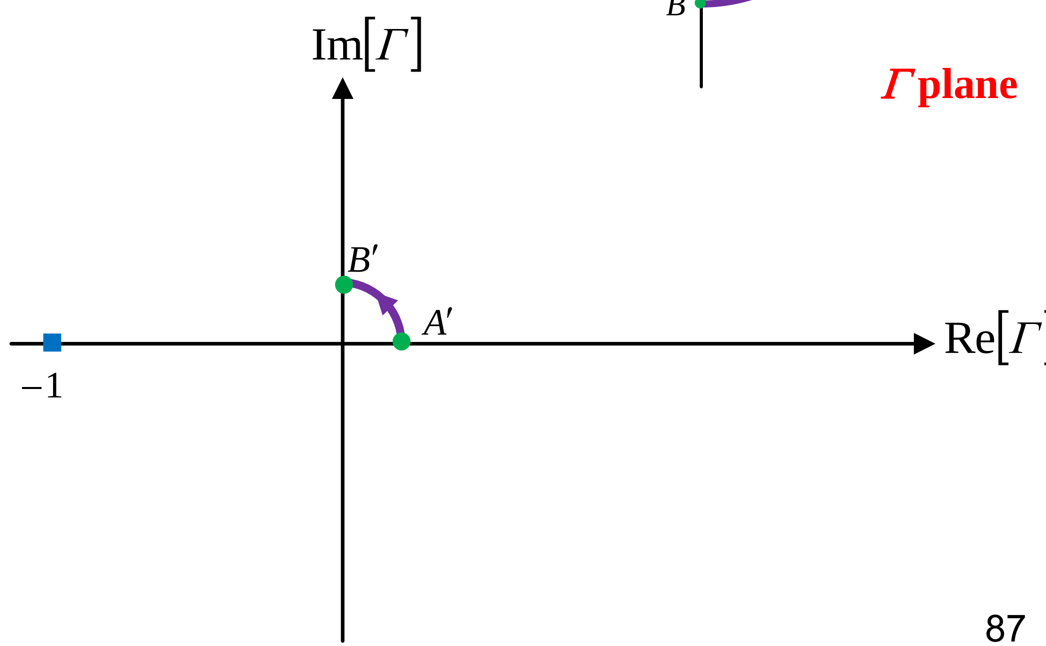
Section AB:

$$s = \lim_{R\to\infty} R\, e^{j\theta}$$

$$\Gamma\left(R e^{j\theta}\right) = \varepsilon\, e^{-j\theta}$$

$A \rightarrow A'$ $\quad \Gamma = \varepsilon\, e^{j0}$

$B \rightarrow B'$ $\quad \Gamma = \varepsilon\, e^{j\pi/2}$



Nyquist Stability Criterion

Solution:

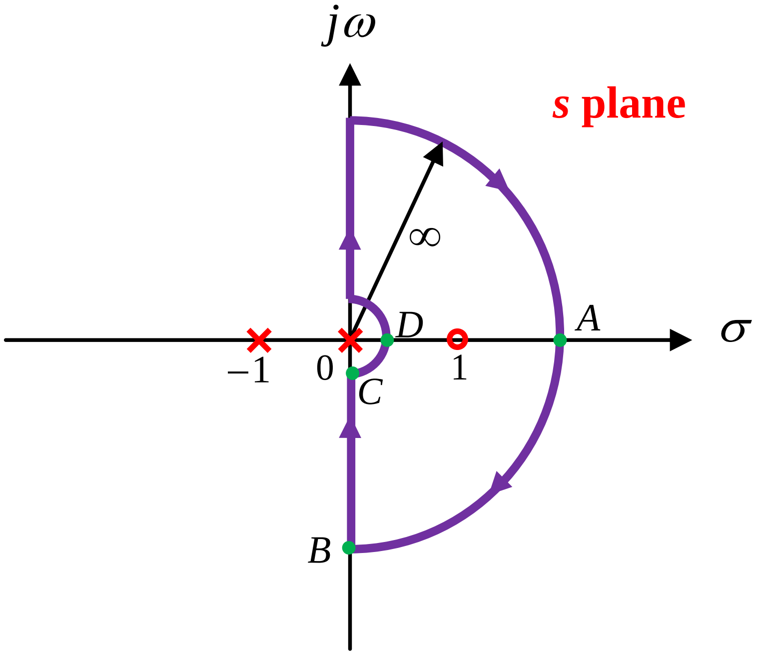
3. **Map** the contour in *s* plane into $\Gamma(s) = G(s)H(s)$.

Section BC: $s = -j\omega$

$$\Gamma(s) = G(s)H(s) = \frac{(s-1)}{s(s+1)}$$

$$\Gamma(j\omega) = \frac{(-j\omega - 1)}{-j\omega(-j\omega + 1)}$$

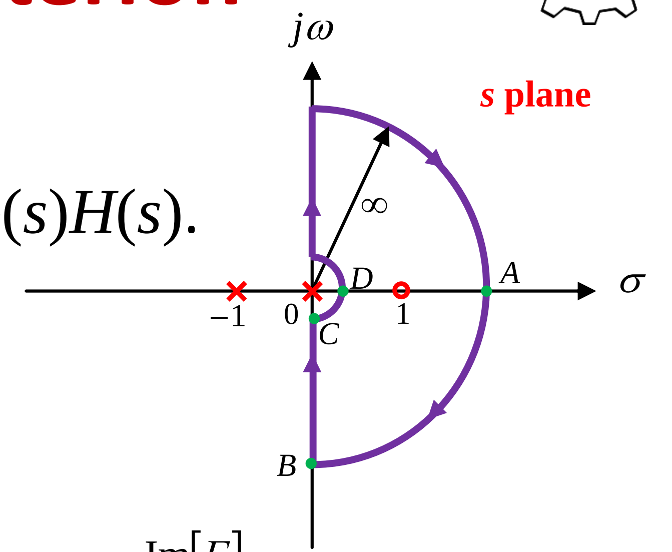
$$\Gamma(j\omega) = \frac{2\omega + j(\omega^2 - 1)}{\omega(\omega^2 + 1)}$$



Nyquist Stability Criterion

Solution:

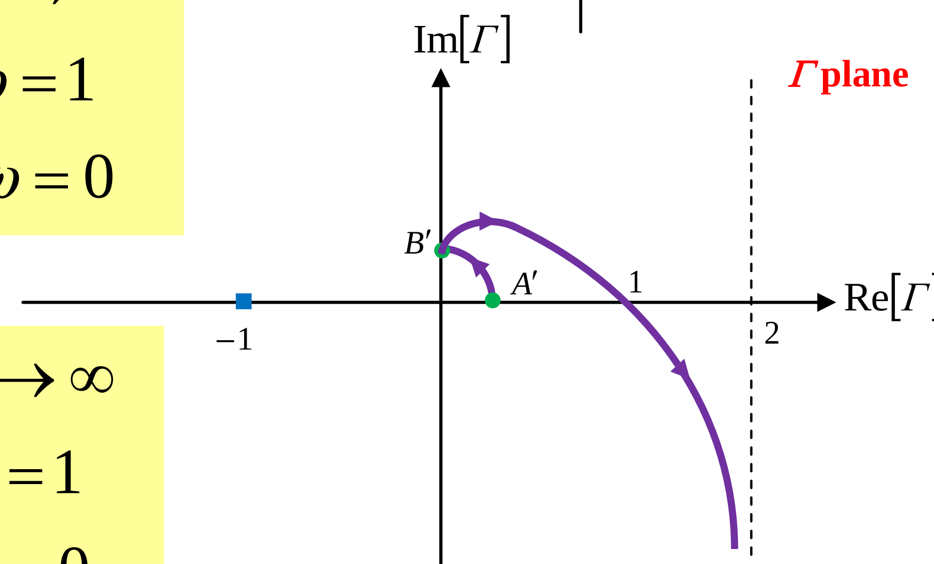
3. **Map** the contour in s plane into $\Gamma(s)=G(s)H(s)$.



Section BC: $s=-j\omega$

$$\mathrm{Re}\left[\Gamma(j\omega)\right]=\frac{2\omega}{\omega\left(\omega^2+1\right)}=\begin{cases}0 & \omega\to\infty\\ 1 & \omega=1\\ 2 & \omega=0\end{cases}$$

$$\mathrm{Im}\left[\Gamma(j\omega)\right]=\frac{\left(\omega^2-1\right)}{\omega\left(\omega^2+1\right)}=\begin{cases}0 & \omega\to\infty\\ 0 & \omega=1\\ -\infty & \omega=0\end{cases}$$



Nyquist Stability Criterion

Solution:

3. **Map** the contour in s plane into $\Gamma(s)=G(s)H(s)$.

Section *CD*:

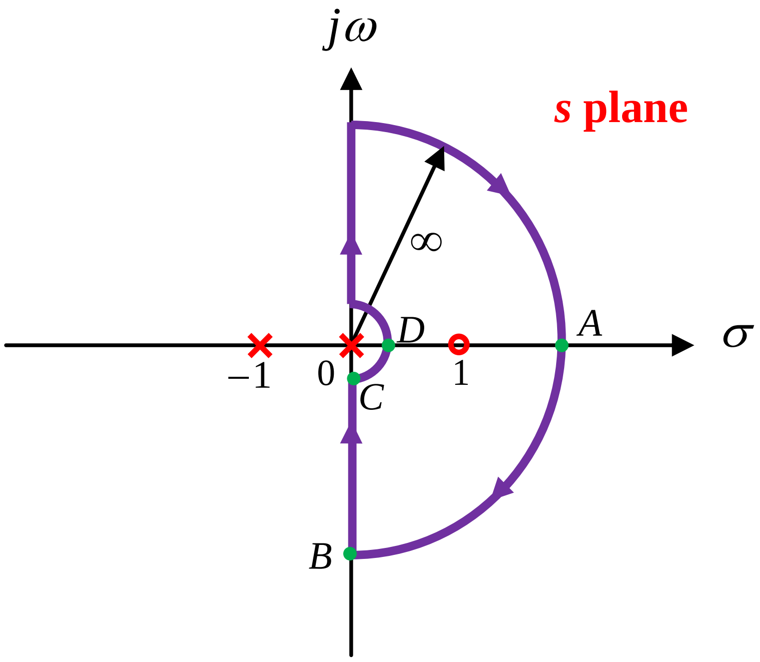
$$s = \underset{\varepsilon \to 0}{\varepsilon}\, e^{j\theta}$$

$$\Gamma(s) = G(s)H(s) = \frac{(s-1)}{s(s+1)}$$

$$\Gamma\left(\varepsilon e^{j\theta}\right) = \frac{\left(\varepsilon e^{j\theta} - 1\right)}{\varepsilon e^{j\theta}\left(\varepsilon e^{j\theta} + 1\right)}$$

$$\Gamma\left(\varepsilon e^{j\theta}\right) = R\, e^{j(\pi-\theta)}$$

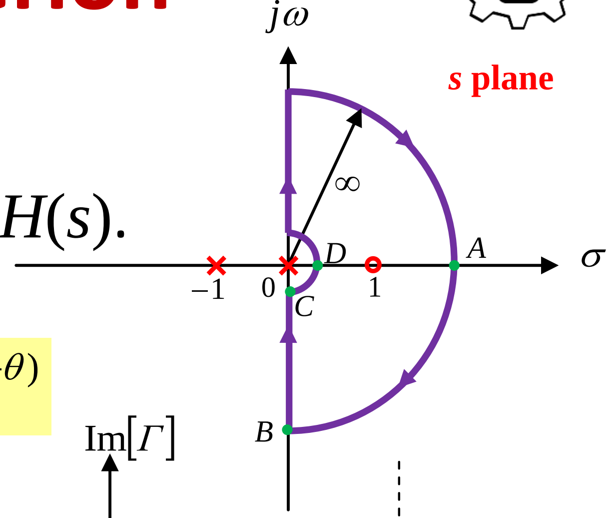
$$-\tfrac{\pi}{2} \le \theta \le 0$$



Nyquist Stability Criterion

Solution:

3. **Map** the contour in s plane into $\Gamma(s) = G(s)H(s)$.



Section CD:

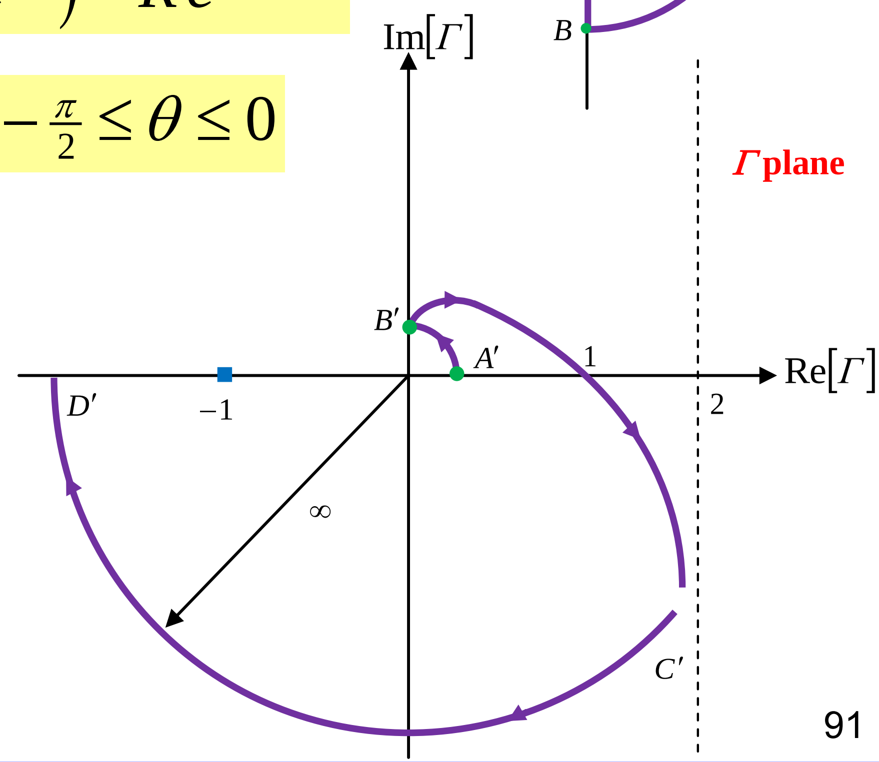
$$s = \lim_{\varepsilon \to 0} \varepsilon \, e^{j\theta}$$

$$\Gamma\left(\varepsilon \, e^{j\theta}\right) = R \, e^{j(\pi - \theta)}$$

$$-\tfrac{\pi}{2} \le \theta \le 0$$

$$C \rightarrow C' \qquad \theta = -\tfrac{\pi}{2} \qquad \Gamma = R\, e^{j3\pi/2}$$

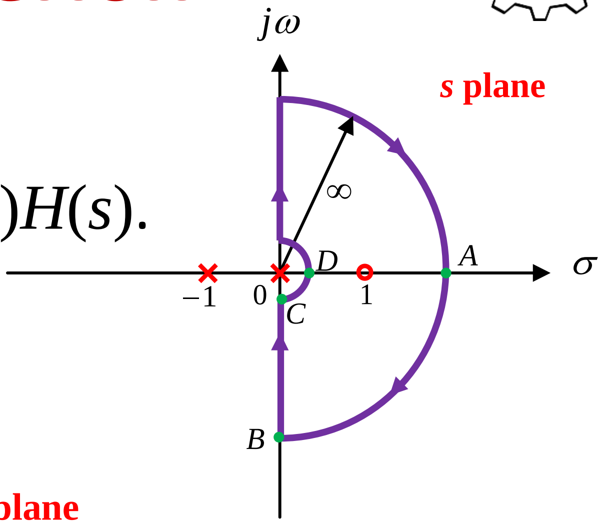
$$D \rightarrow D' \qquad \theta = 0 \qquad \Gamma = R\, e^{j\pi}$$

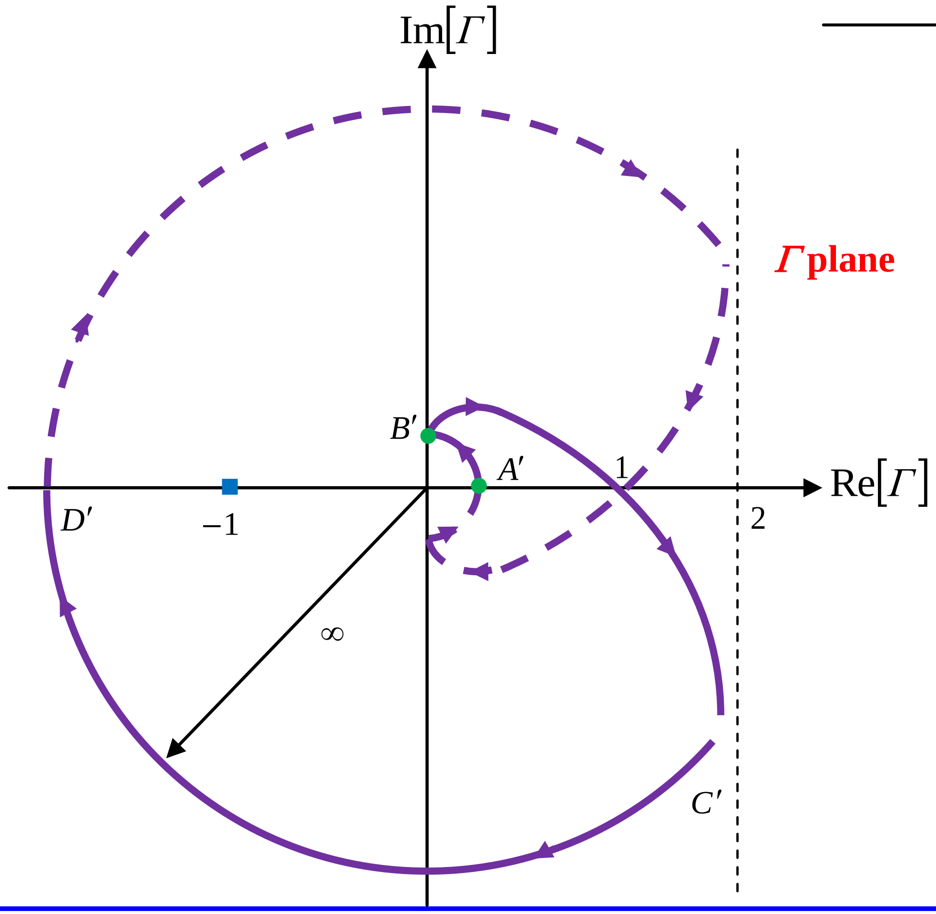


Nyquist Stability Criterion

Solution:

3. **Map** the contour in s plane into $\Gamma(s)=G(s)H(s)$.

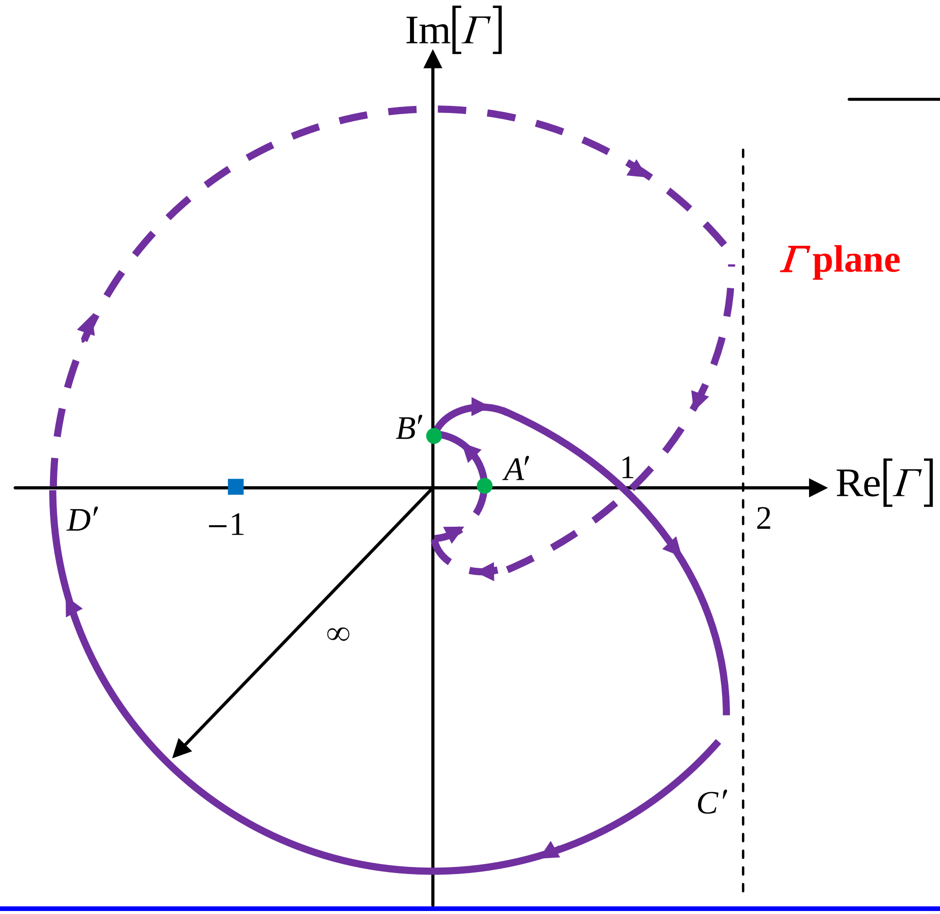


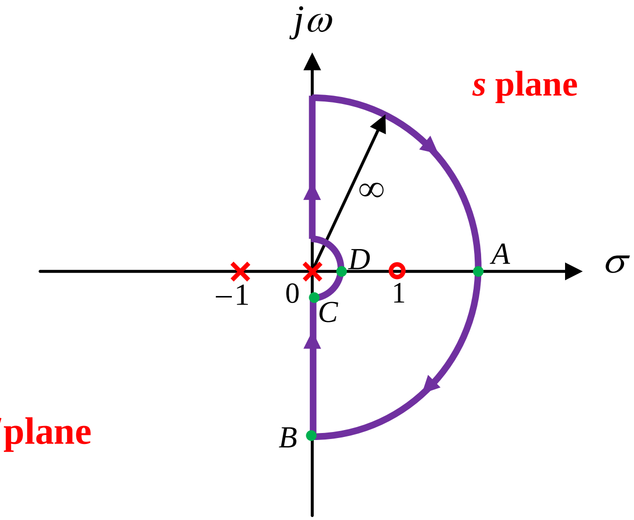


Nyquist Stability Criterion

Solution:

4. Find the number of poles of $G(s)H(s)$ in the right-half s plane, i.e. $\boldsymbol{P}$.



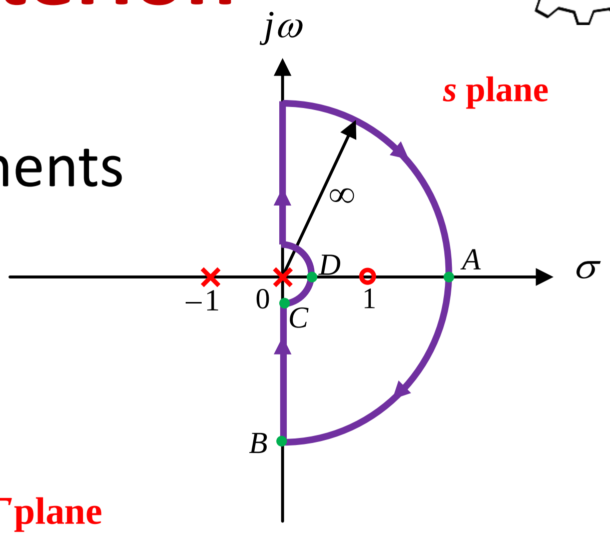


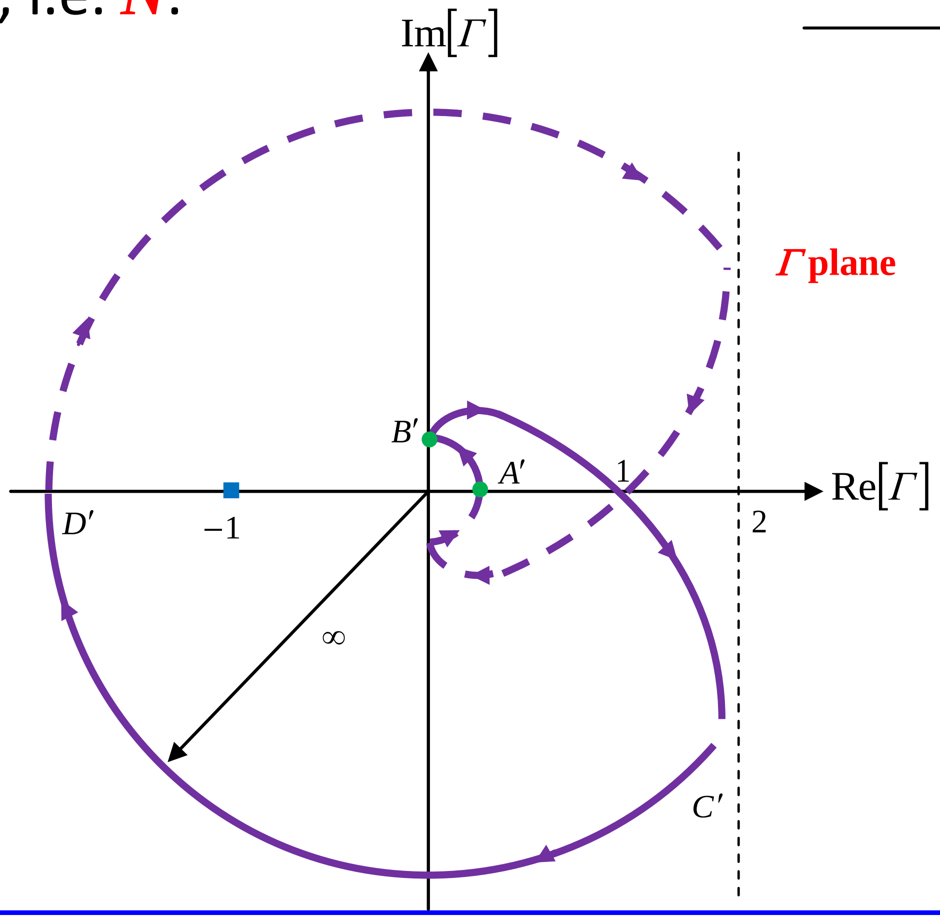
$P = 0$

Nyquist Stability Criterion

Solution:

5. Count the number of clockwise encirclements of -1 point, i.e. N.



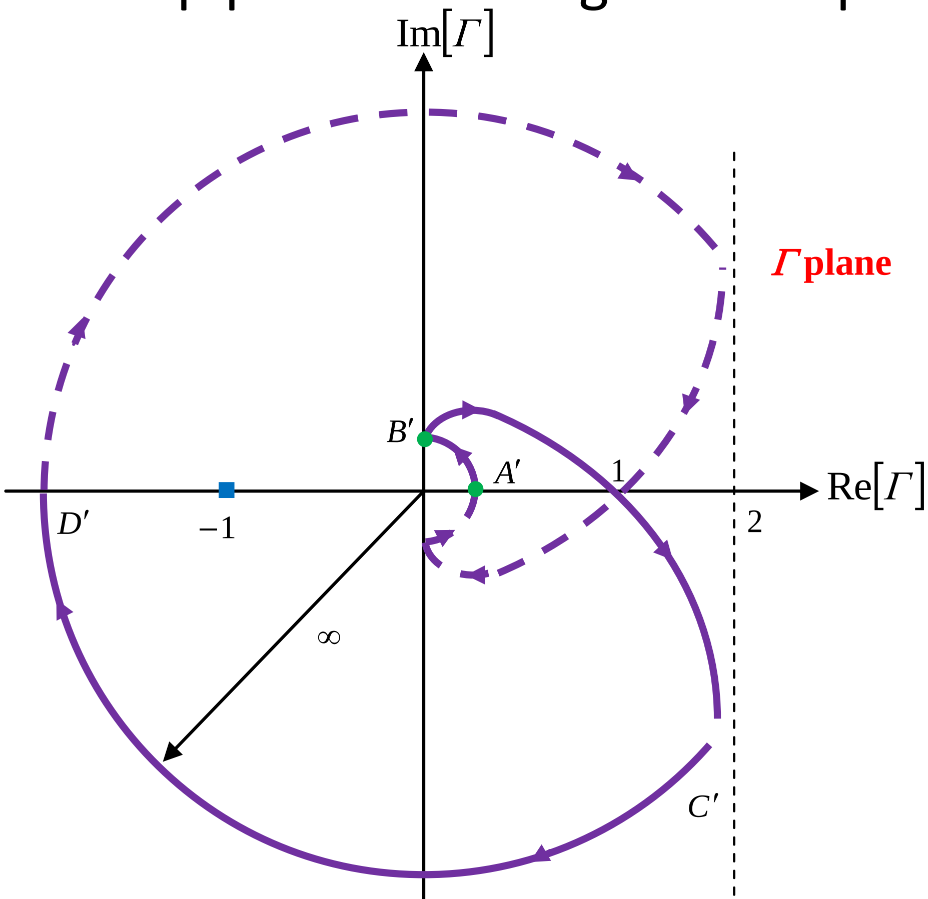


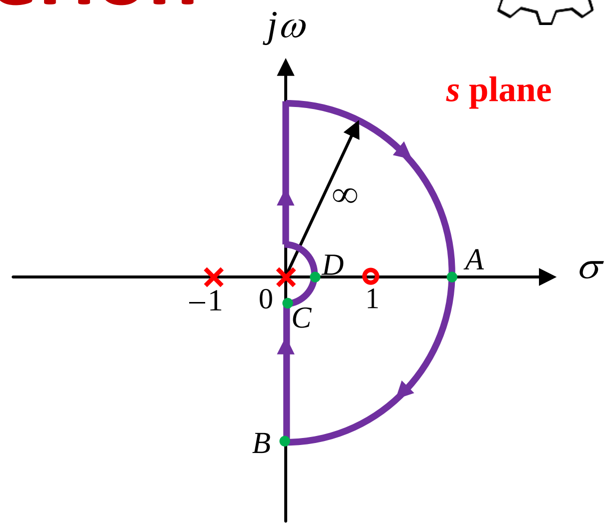
$$N = 1$$

Nyquist Stability Criterion

Solution:

6. Find $Z = N + P$ which is the number of closed-loop poles in the right-half s plane.



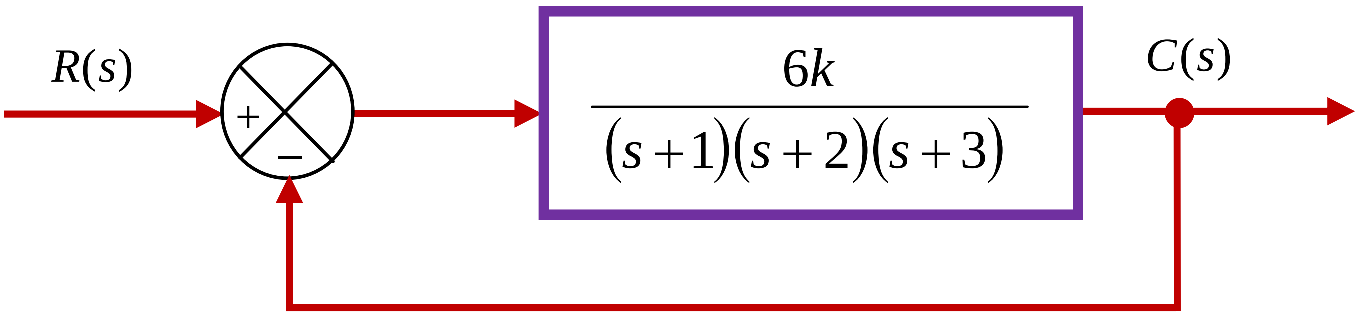


$Z = 1$

The system is unstable.

Nyquist Stability Criterion

- **Example**: Using Nyquist stability criterion find the range of positive k in which the following system is stable



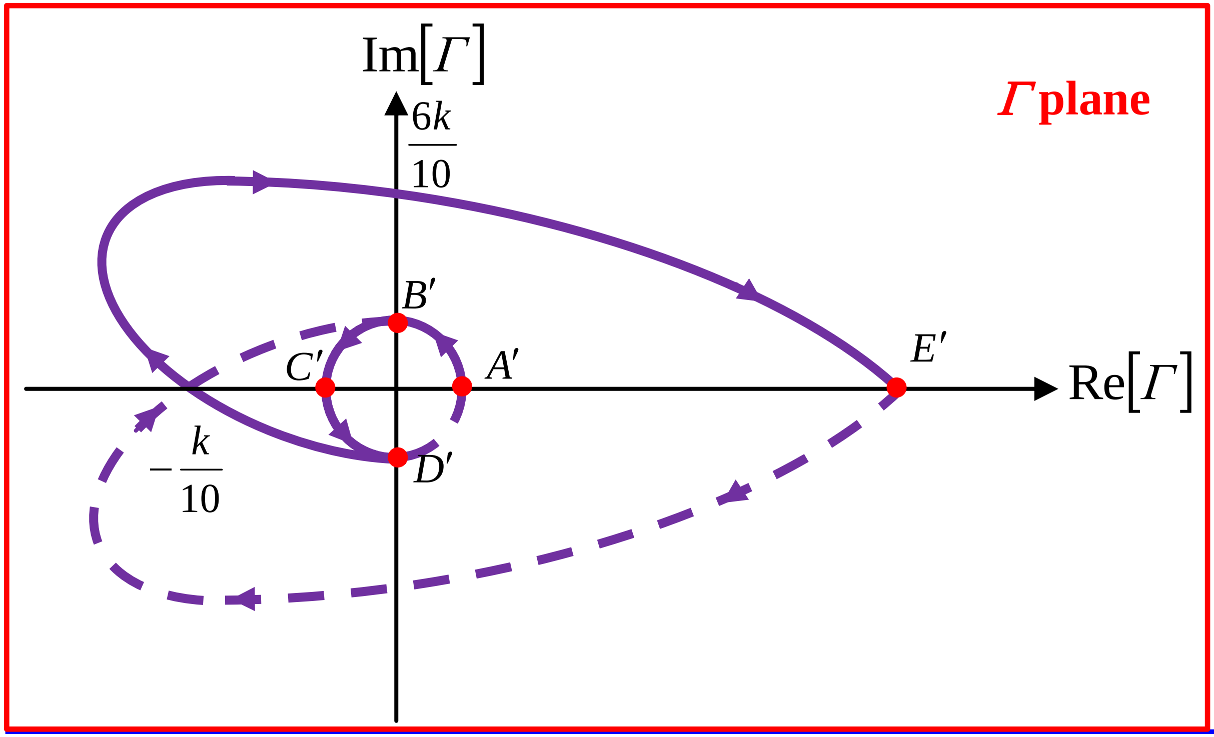
Nyquist Stability Criterion

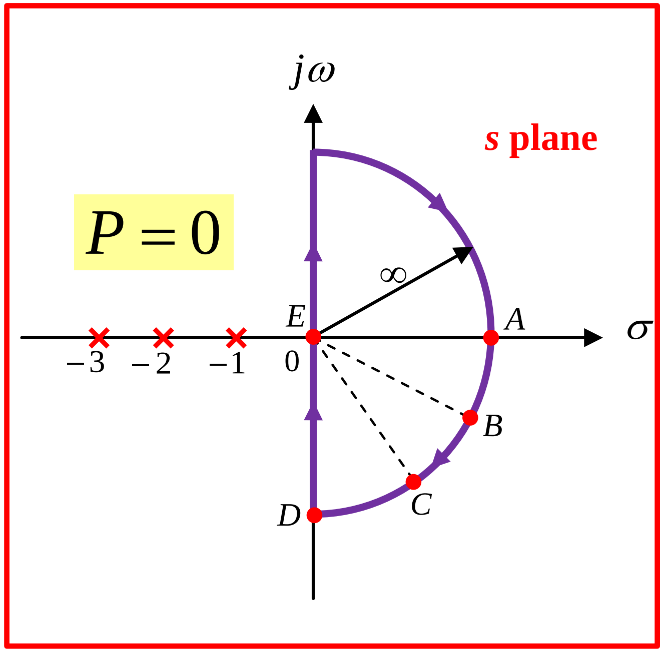
Solution:

If $k > 10$ → $N = 2$ → $Z = 2$ → Unstable system

If $k = 10$ → Critically stable system

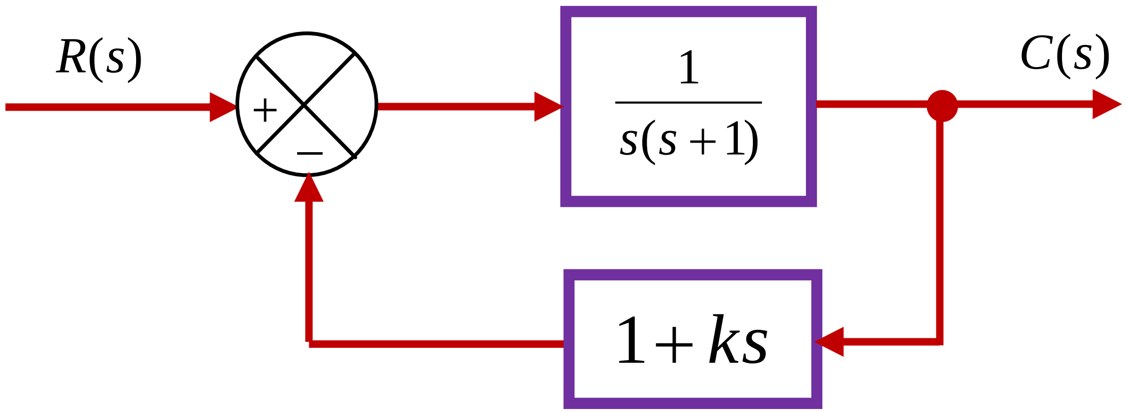
If $k < 10$ → $N = 0$ → $Z = 0$ → Stable system





Nyquist Stability Criterion

- **Example**: Using Nyquist stability criterion find the range of positive k in which the following system is stable



Nyquist Stability Criterion

- **Solution**: The characteristic equation is expressed as

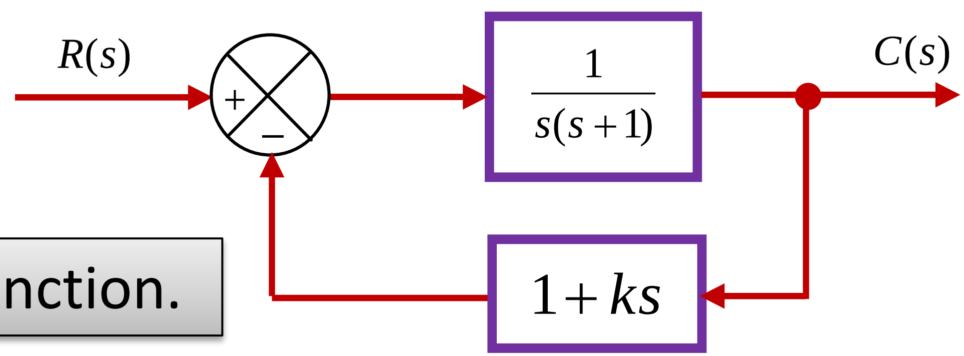
$$\Delta(s)=1+\frac{1+ks}{s(s+1)}=0 \quad \Rightarrow \quad \frac{s^2+s+1+ks}{s(s+1)}=0 \quad \Rightarrow$$

$$s^2+s+1+ks=0$$

Divide by the parts without k

$$\Rightarrow \quad 1+\frac{ks}{s^2+s+1}=0 \quad \Rightarrow$$

$$G(s)H(s)=\frac{ks}{s^2+s+1}$$



It is the virtual loop transfer function.

Nyquist Stability Criterion

Important note:

- To investigate the stability of system with a variable, e.g. k, using Nyquist stability criterion, the variable should be as a gain in the loop transfer function.

$$G(s)H(s) = k\frac{N(s)}{D(s)}$$

- If it is not the case, the **virtual** loop transfer function should be formed.

Phase Margin & Gain Margin

1. Gain Margin (GM):

- Assume ω_p is the frequency in which $\angle GH(j\omega_p) = -180$

ω_p is called phase crossover frequency.

- The gain margin is obtained as $GM = \dfrac{1}{\left|GH(j\omega_p)\right|}$

- Or in the case of dB it is $GM_{dB} = -\left|GH(j\omega_p)\right|_{dB}$

Phase Margin & Gain Margin

2. Phase Margin (PM):

- Assume ω_g is the frequency in which $|GH(j\omega_g)| = 1$ or $|GH(j\omega_g)|_{dB} = 0$

ω_g is called gain crossover frequency.

- The phase margin is obtained as $PM = 180 + \angle GH(j\omega_g)$

Phase and gain margins are useful in minimum-phase systems.

Phase Margin & Gain Margin

In a **minimum-phase** system to have **stability** both **phase margin** and **gain margin in dB** should be **positive**. i.e.

$$PM = 180 + \angle GH(j\omega_g) > 0 \quad \Rightarrow \quad -180 < \angle GH(j\omega_g) < 0$$

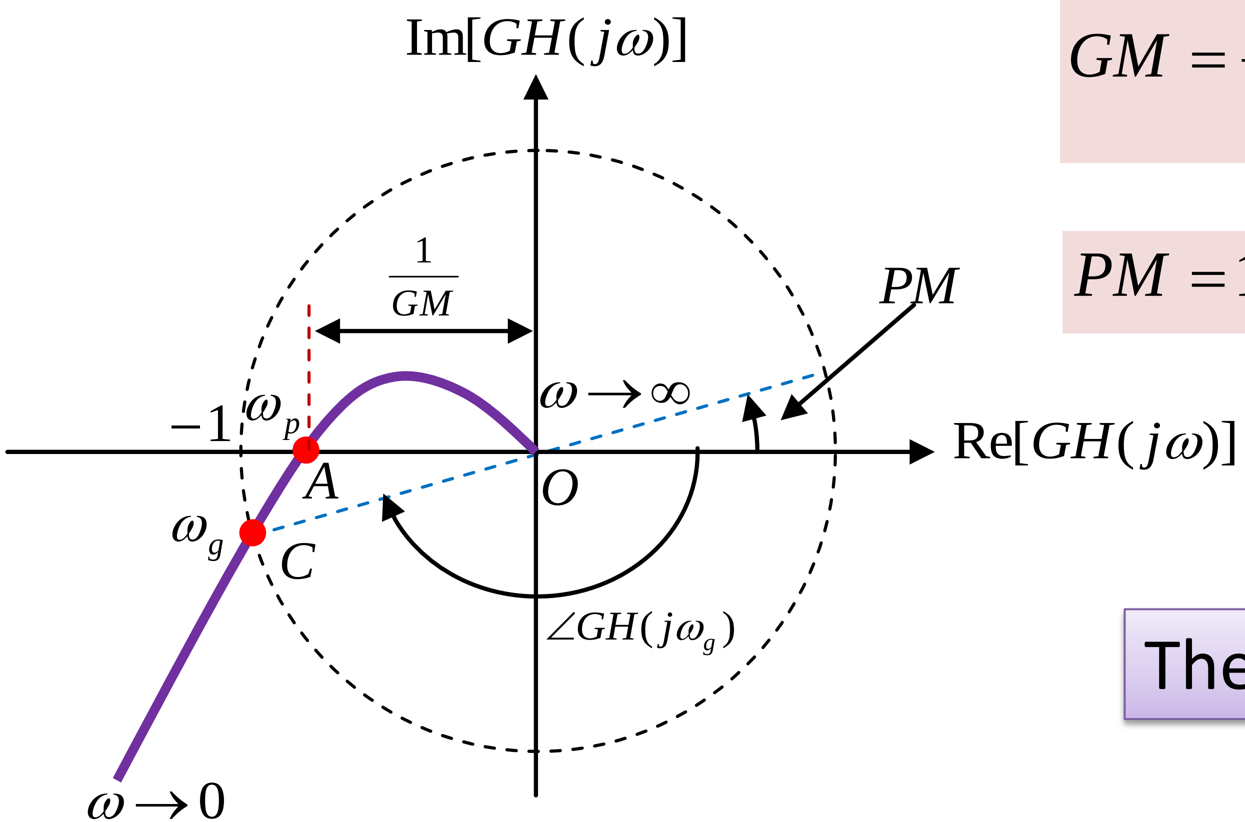
and

$$GM_{dB} = -\left|GH(j\omega_p)\right|_{dB} > 0 \quad \Rightarrow \quad GM = \left|GH(j\omega_p)\right| < 1$$

Note that phase and gain margins cannot be used for stability analysis in non-minimum-phase systems.

Phase Margin & Gain Margin in Polar Diagram

Consider a minimum-phase system with the following polar diagram



$$GM = \frac{1}{|OA|} > 1$$

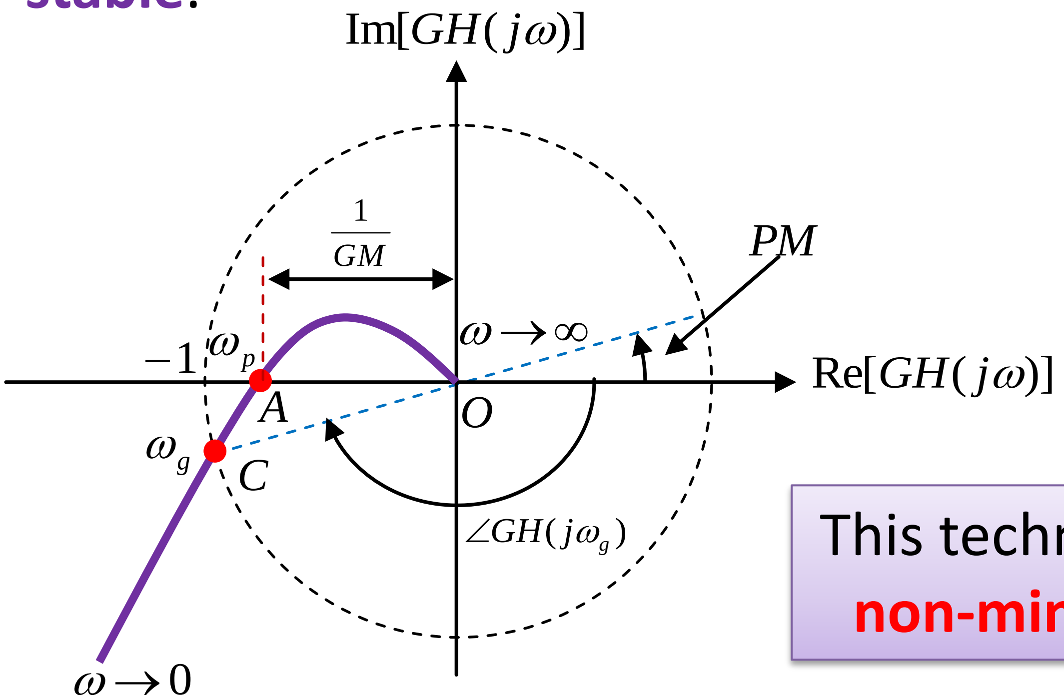
$$GM_{dB} > 0$$

$$PM = 180 + \angle GH(j\omega_g) > 0$$

The system is stable.

Phase Margin & Gain Margin in Polar Diagram

In polar diagram of **minimum-phase** systems, moving from zero frequency to infinity frequency, if point **-1 is located on the left** side of the trajectory (from zero to infinity frequency), the system is **stable**.

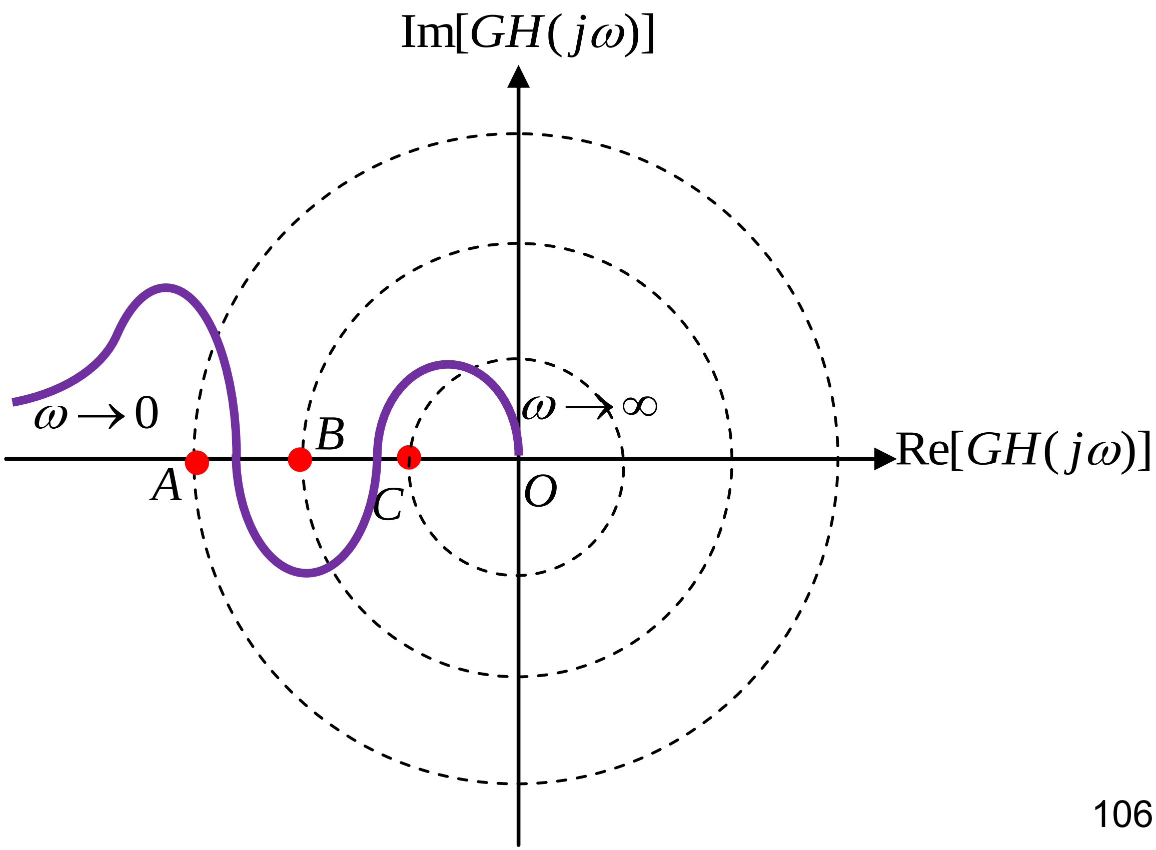


This technique **cannot** be used in **non-minimum-phase** systems.

Phase Margin & Gain Margin in Polar Diagram

Example: Consider the following polar diagram of a minimum-phase system. Discuss on the stability if

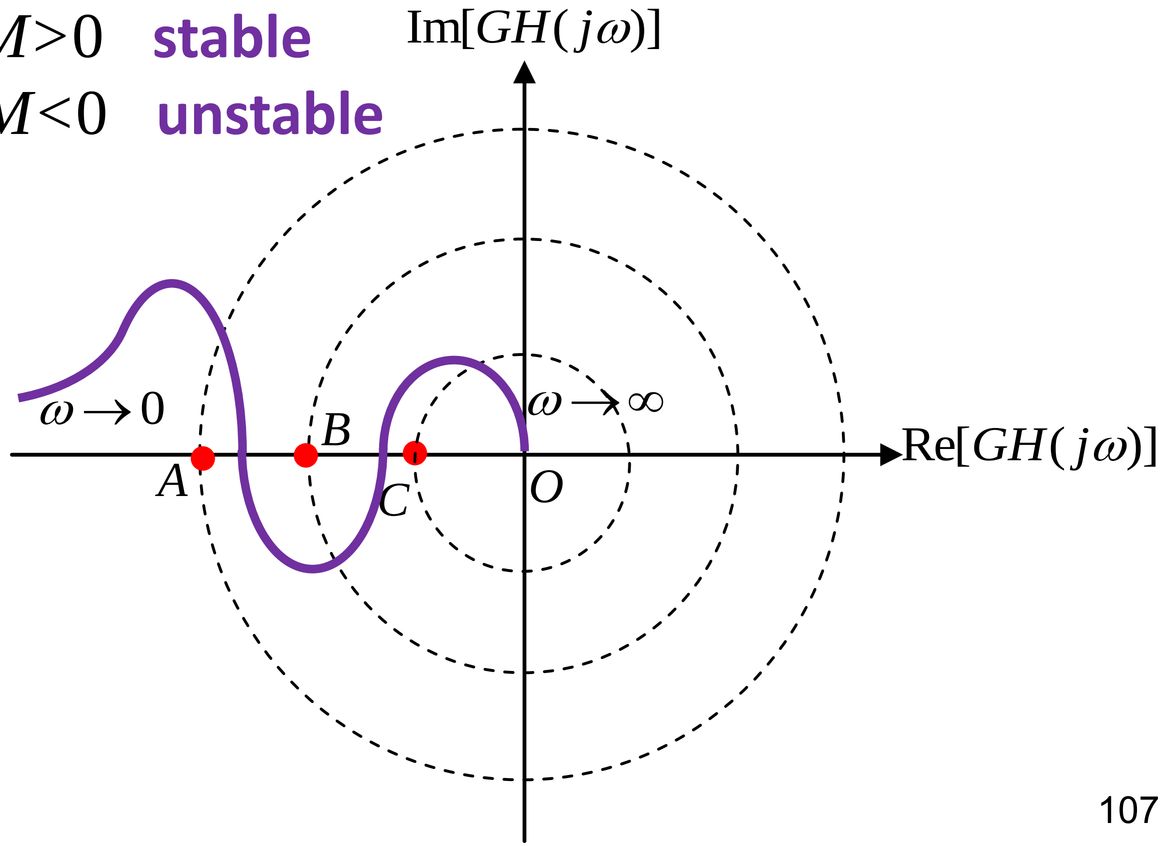
1) Point -1 is at point A
2) Point -1 is at point B
3) Point -1 is at point C



Phase Margin & Gain Margin in Polar Diagram

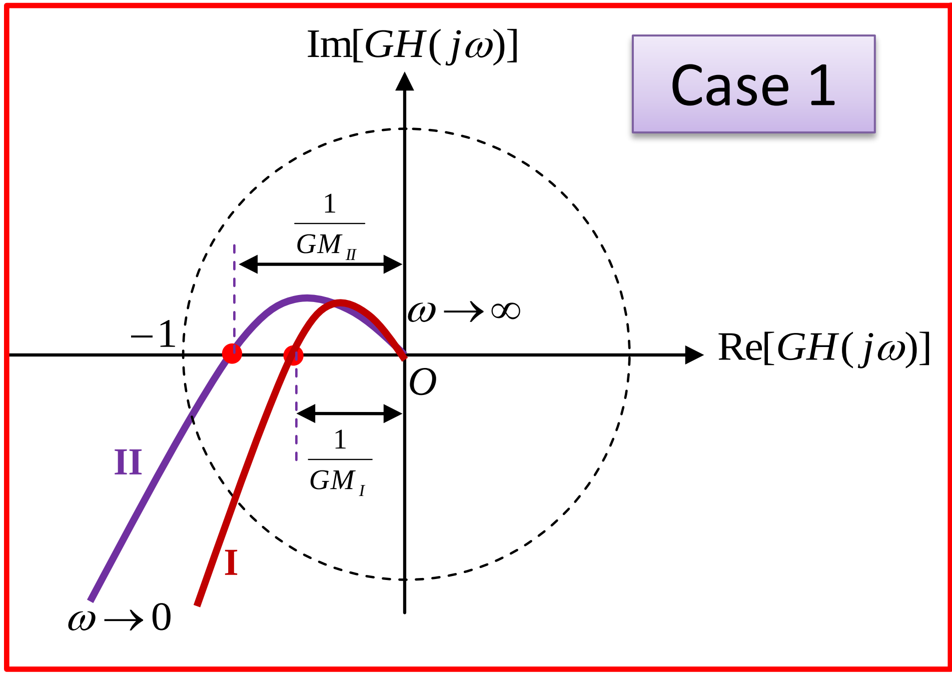
Solution:

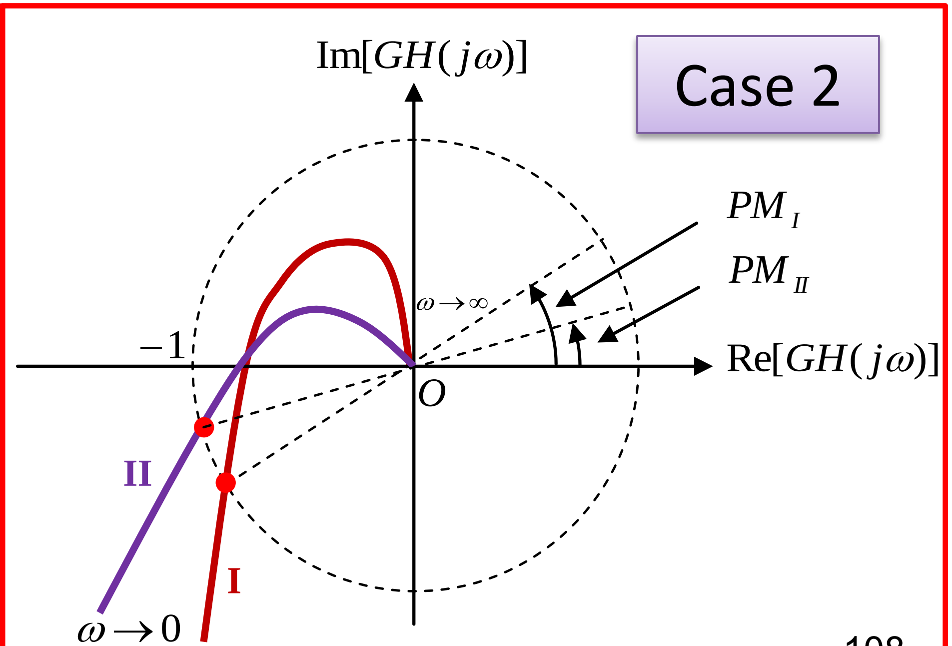
1. A=-1 $PM<0$ **unstable**
2. B=-1 $PM>0$ & $GM>0$ **stable**
3. C=-1 $PM<0$ & $GM<0$ **unstable**



Relative Stability using Phase Margin & Gain Margin

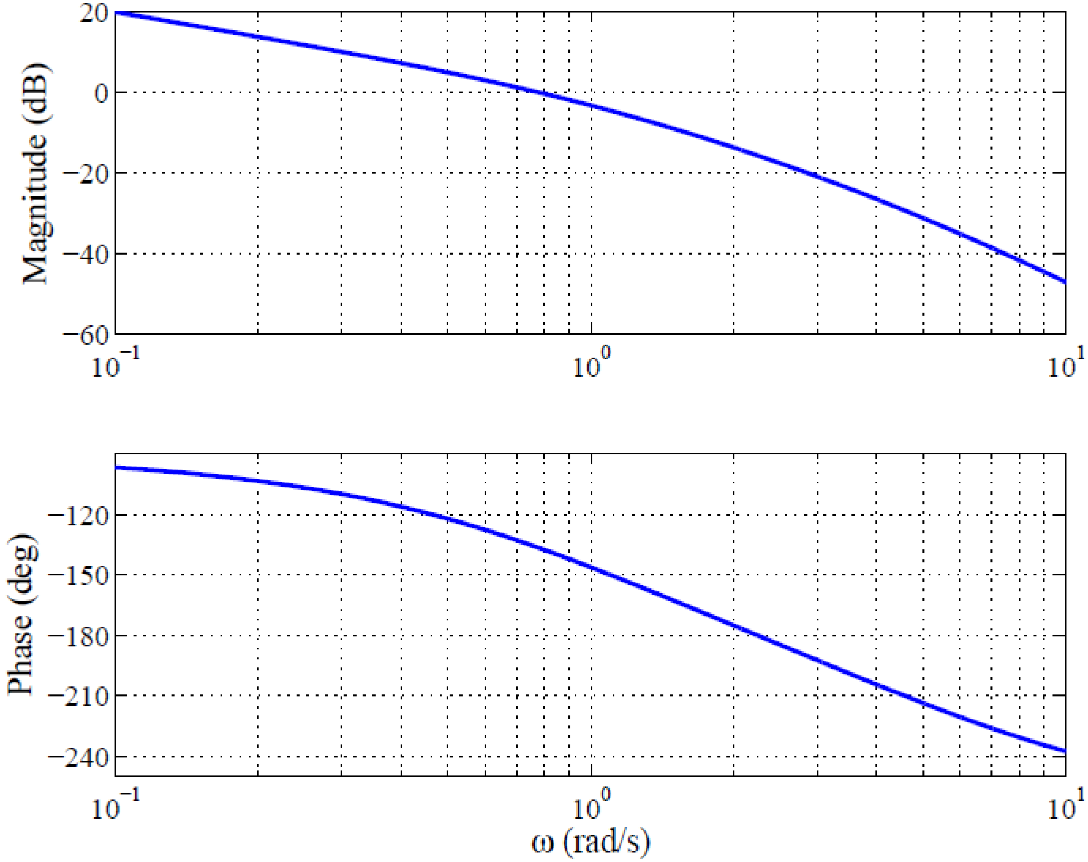
- Comparing two **stable** **minimum-phase** systems, the one having **higher gain margin** or in the case of **equal gain margins**, the one having **higher phase margin** is **more stable**.
- In the following examples system I is more stable.





Phase Margin & Gain Margin in Bode Diagrams

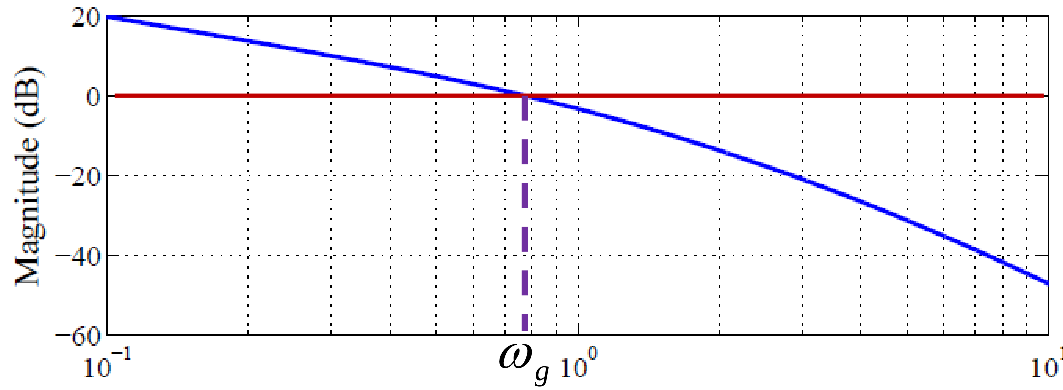
Example: Calculate the gain and phase margins from the following Bode diagrams



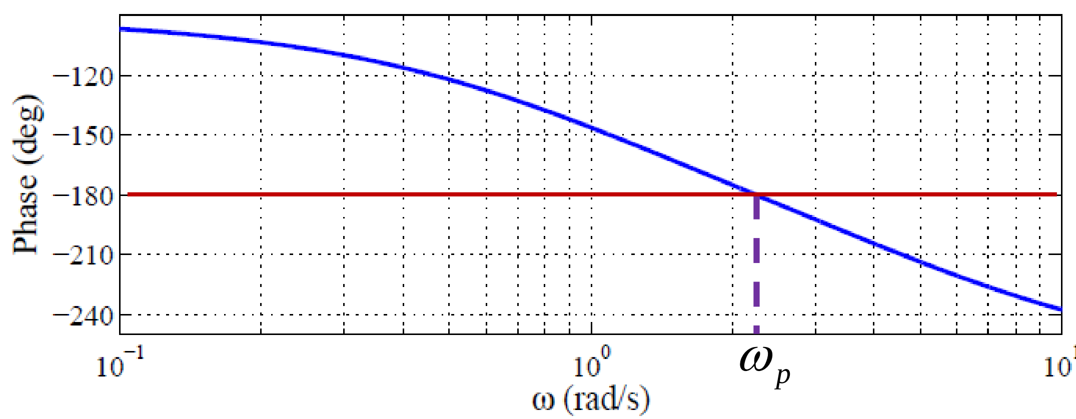
Phase Margin & Gain Margin in Bode Diagrams

Solution: Find the phase and gain crossover frequency (ω_p and ω_g)

$$\omega_g = 0.78 \ \ \text{rad/s}$$



$$\omega_p = 2.2 \ \ \text{rad/s}$$

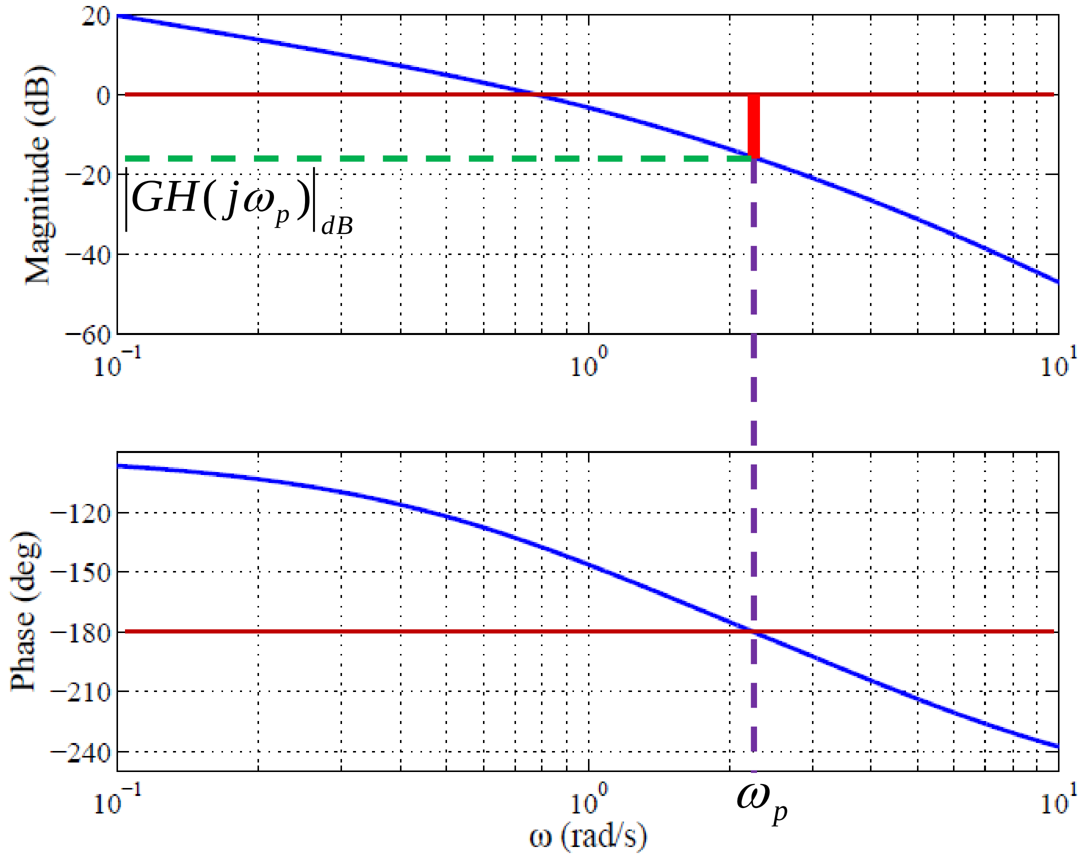


Phase Margin & Gain Margin in Bode Diagrams

Solution: Find the gain margin

$$\omega_p = 2.2 \ \text{rad/s}$$

$$GM_{dB} = -\left|GH(j\omega_p)\right|_{dB}$$
$$= -(-16) = 16 \text{ dB}$$

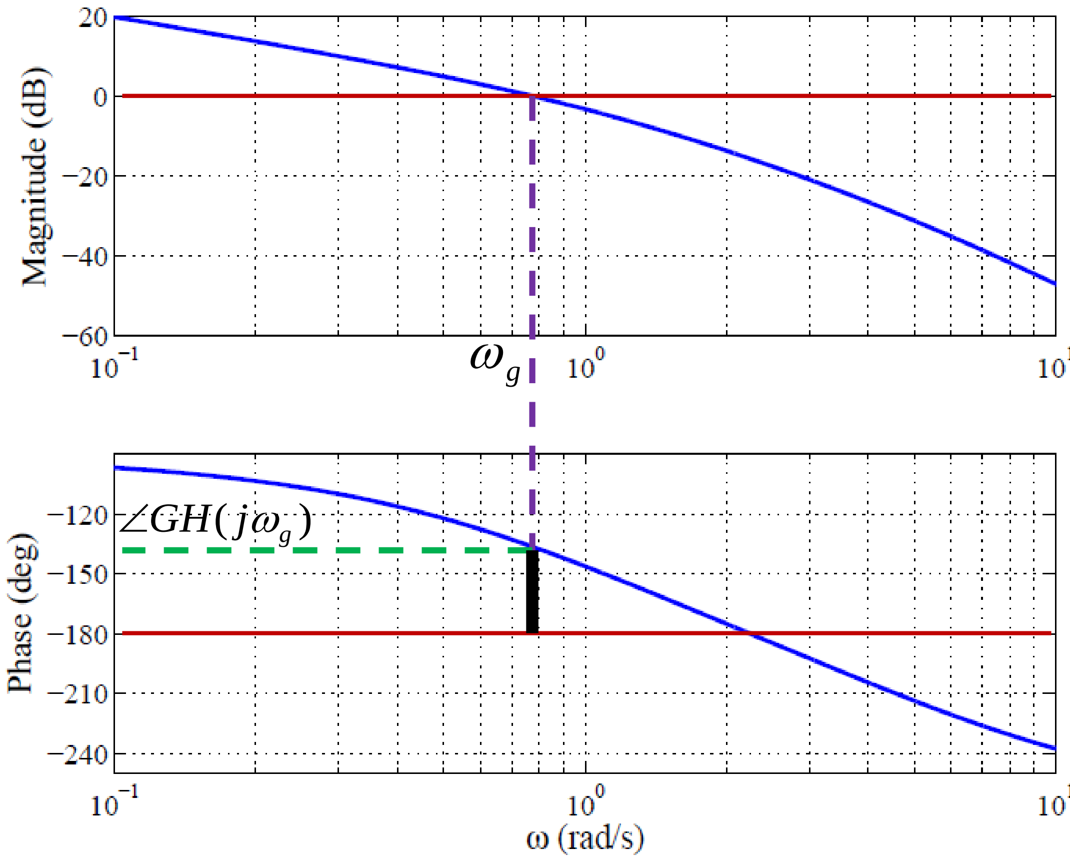


Phase Margin & Gain Margin in Bode Diagrams

Solution: Find the phase margin

$$\omega_g = 0.78 \text{ rad/s}$$

$$PM = 180 + \angle GH(j\omega_g)$$
$$= 180 - 137$$
$$= 43^\circ$$



A Few Points on Phase Margin & Gain Margin

1. **Gain margin** of **first- and second-order** systems is **infinity** since Bode phase diagram never reaches -180 degrees.
2. **Non-minimum-phase** system with negative phase margin and/or negative gain margin MAY be stable.
3. In **minimum-phase** systems with **several** phase and/or gain margins, only one positive phase margin and one positive gain margin leads to stability.
4. In practice for **good stability** PM > 45 degrees and GM > 6 dB.