In The Name of God The Most Compassionate, The Most Merciful

General Theory of Electric Machines

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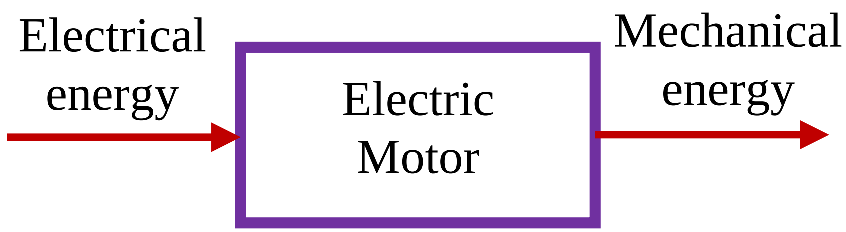
Chapter 4
Induction Machines

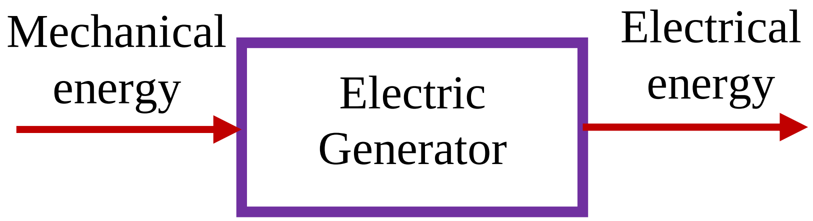
Induction Motors and Generators

An induction machine can be used as a motor or a generator.

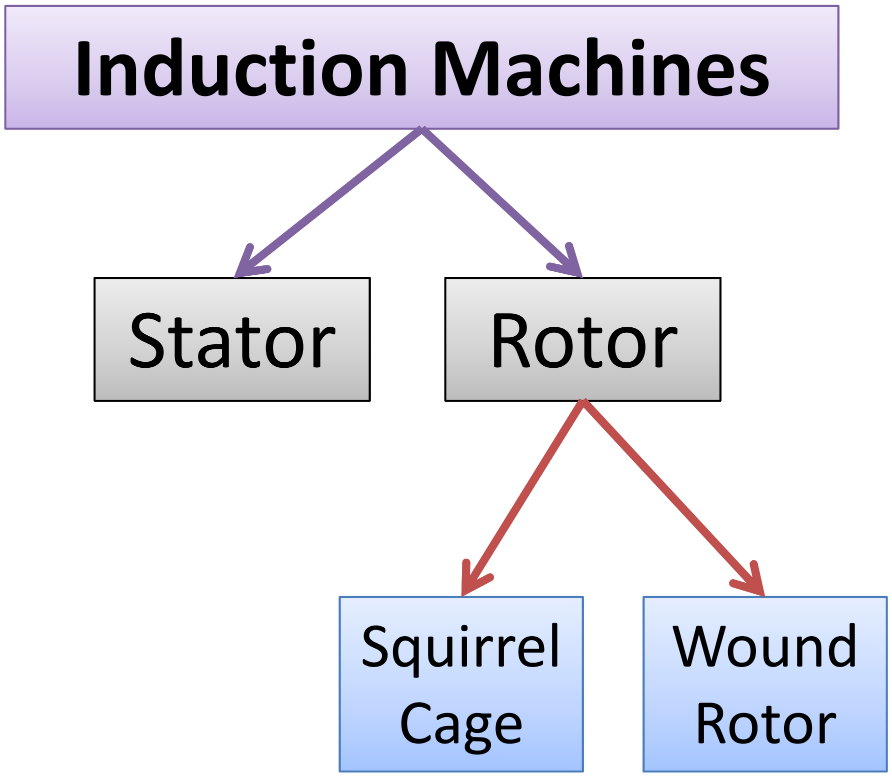
- **Electric motors** receive electrical energy as input and provide mechanical energy as output.

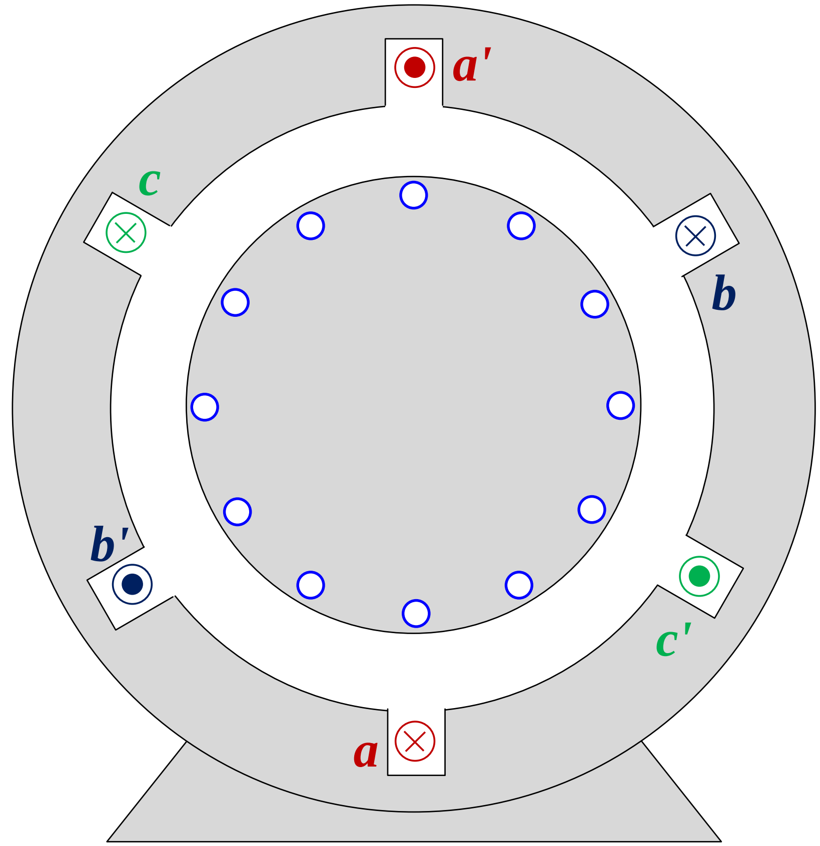


- **Electric generators** receive mechanical energy as input and provide electrical energy as output.



Structure of Induction Machines

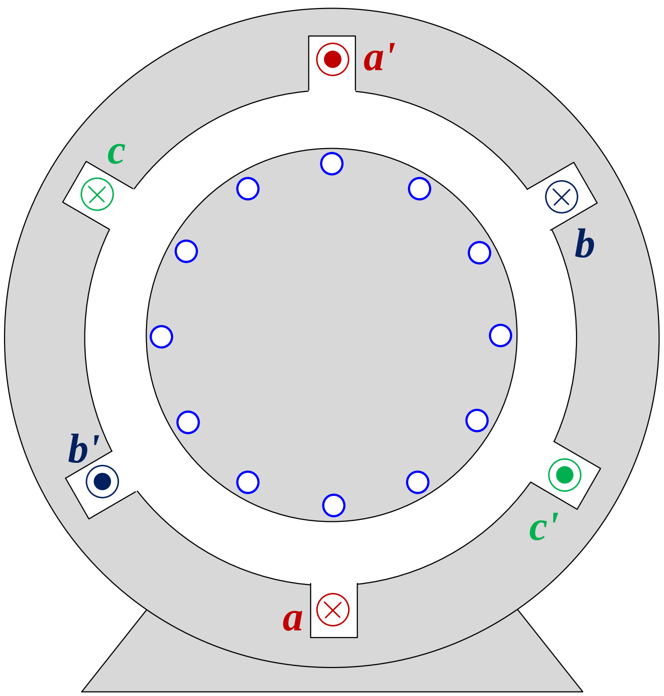


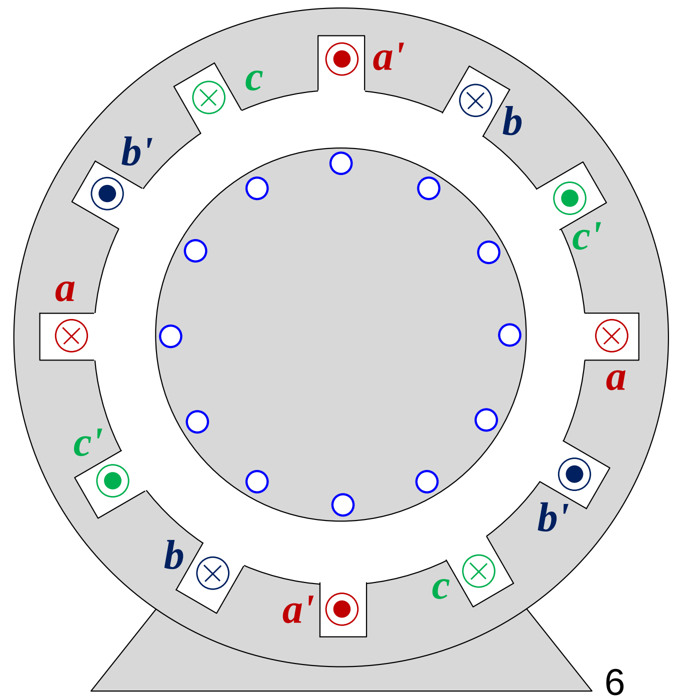


Structure of Induction Machines

1. Stator

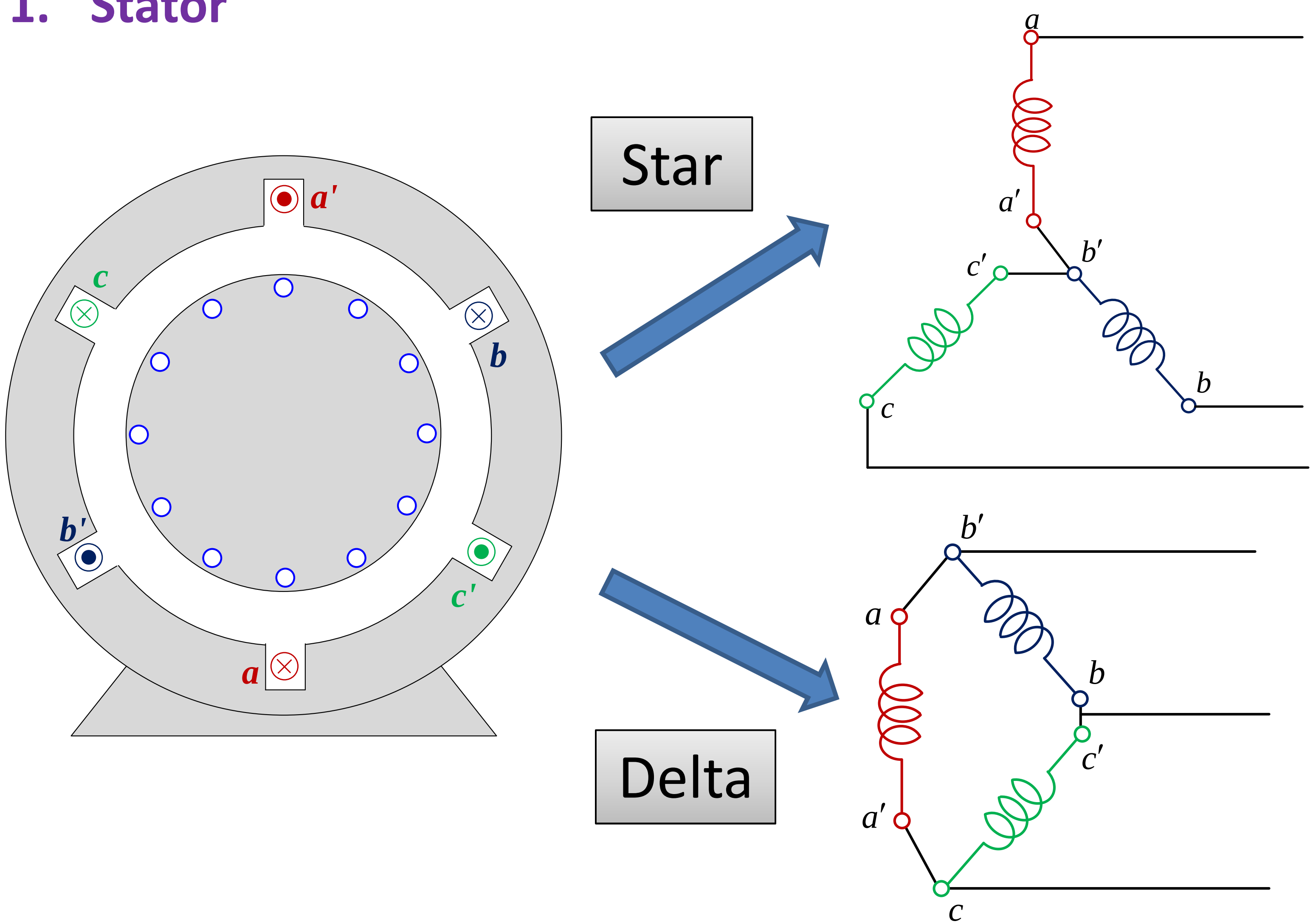
- The stator core is made from the **laminated** steel.
- The armature windings are located in stator **slots**.
- Three-phase windings are connected in **star** or **delta** configuration.





Structure of Induction Machines

1. Stator

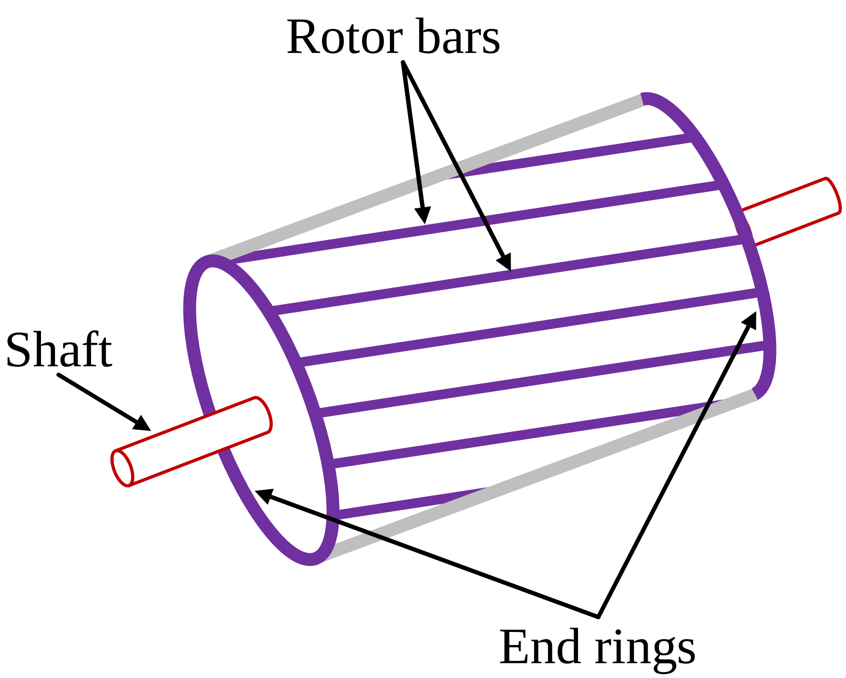


Structure of Induction Machines

2. Rotor

A. Squirrel Cage

- Aluminium or copper bars are located in the rotor **slots**.
- The bars are short-circuited from both sides using **end rings**.

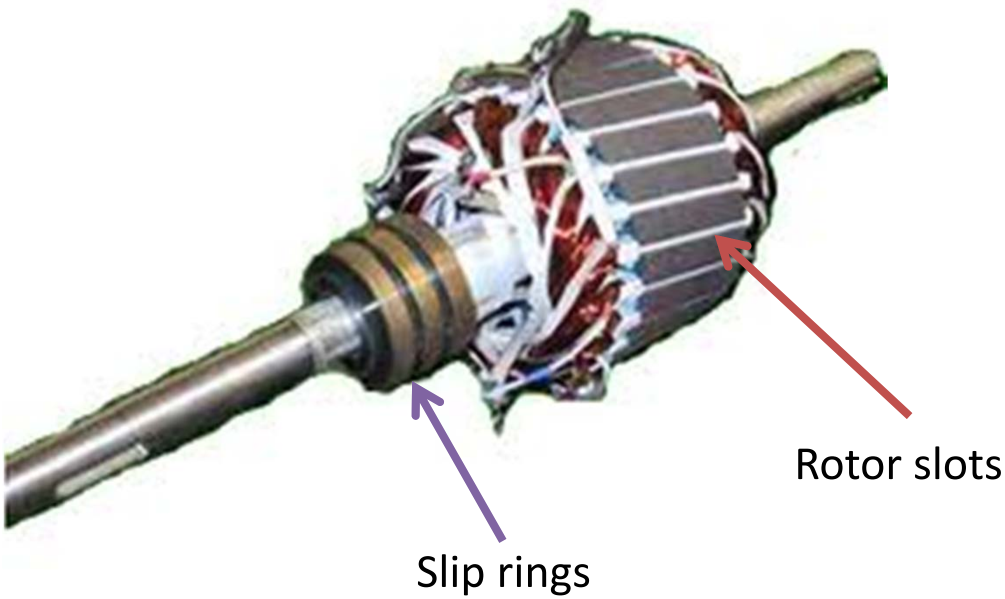


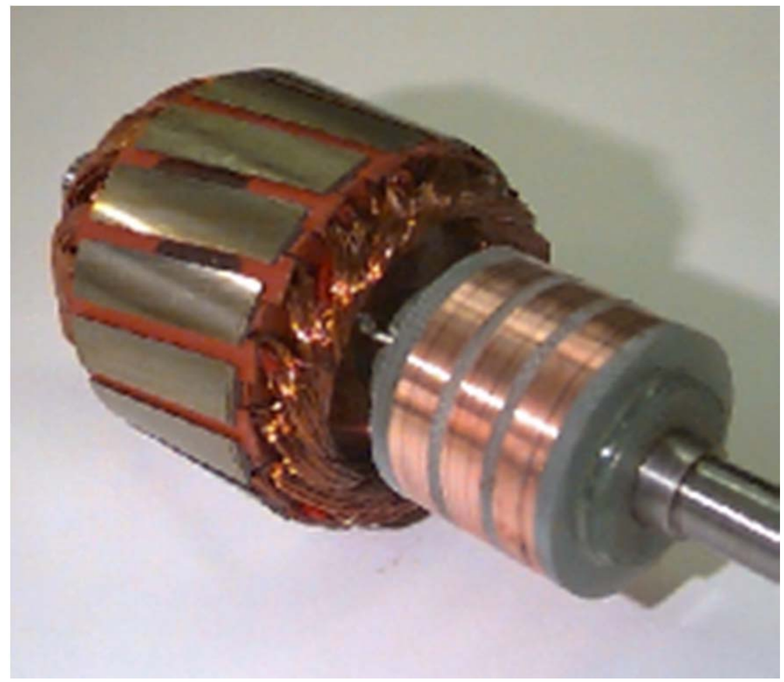
Structure of Induction Machines

2. Rotor

B. Wound Rotor

- Aluminium or copper windings are located in the rotor **slots**.
- There are three **slip-rings** and brushes used for energy transfer.





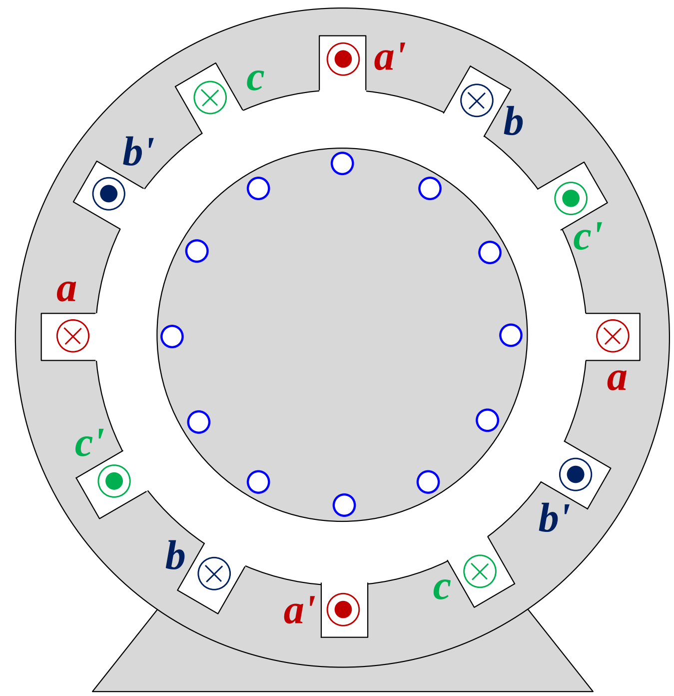
Basic Notions of Induction Machines

1. **Synchronous speed.** Assume the frequency of the applied voltage to the stator winding is f and the machine has P poles; the synchronous speed is defined as:

$$N_s = \frac{120 f}{P} \quad \text{rpm}$$

$$n_s = \frac{2 f}{P} \quad \text{rps}$$

$$\omega_s = \frac{4\pi f}{P} \quad \text{rad/s}$$



Synchronous speed is a mechanical quantity.

Basic Notions of Induction Machines

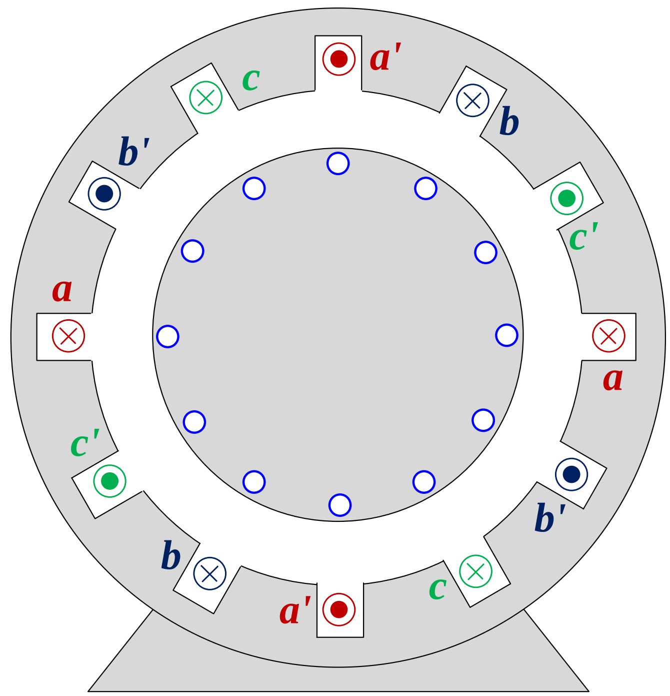
2. **Rotor speed** is the speed of the rotor.

In motoring mode

$N_r < N_s$ rpm

$\omega_r < \omega_s$ rad/s

$n_r < n_s$ rps



In generating mode

$N_r > N_s$ rpm

$\omega_r > \omega_s$ rad/s

$n_r > n_s$ rps

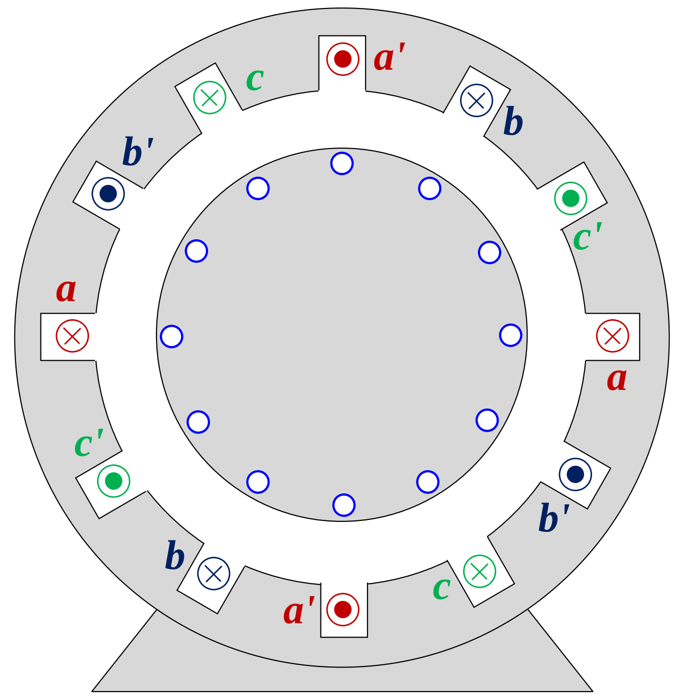
Rotor speed is a mechanical quantity.

3. **Slip speed** is the difference between the synchronous speed and the rotor speed:

$$N_{slip} = N_s - N_r \quad \text{rpm}$$

$$n_{slip} = n_s - n_r \quad \text{rps}$$

$$\omega_{slip} = \omega_s - \omega_r \quad \text{rad/s}$$



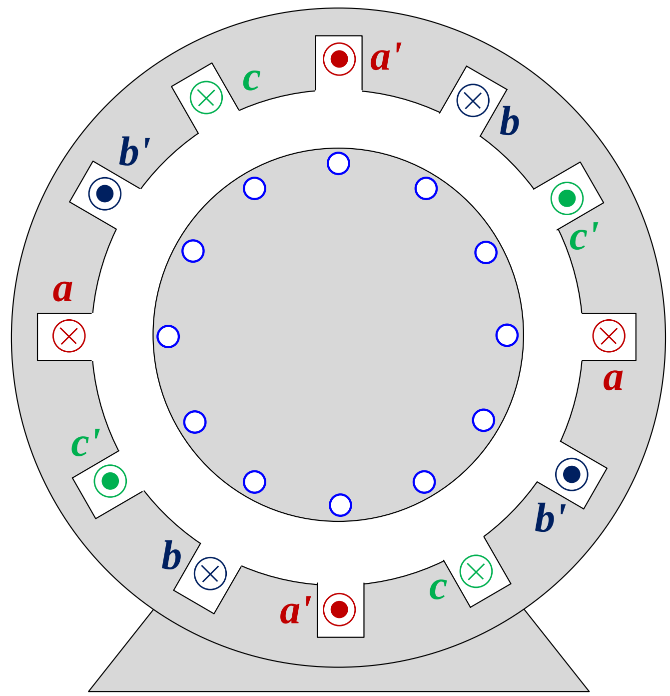
Slip speed is a mechanical quantity.

Basic Notions of Induction Machines

4. **Slip** is the slip speed divided by the synchronous speed:

$$S = \frac{N_s - N_r}{N_s} = \frac{n_s - n_r}{n_s} = \frac{\omega_s - \omega_r}{\omega_s}$$

$$N_r = (1 - S) N_s \quad \text{rpm}$$



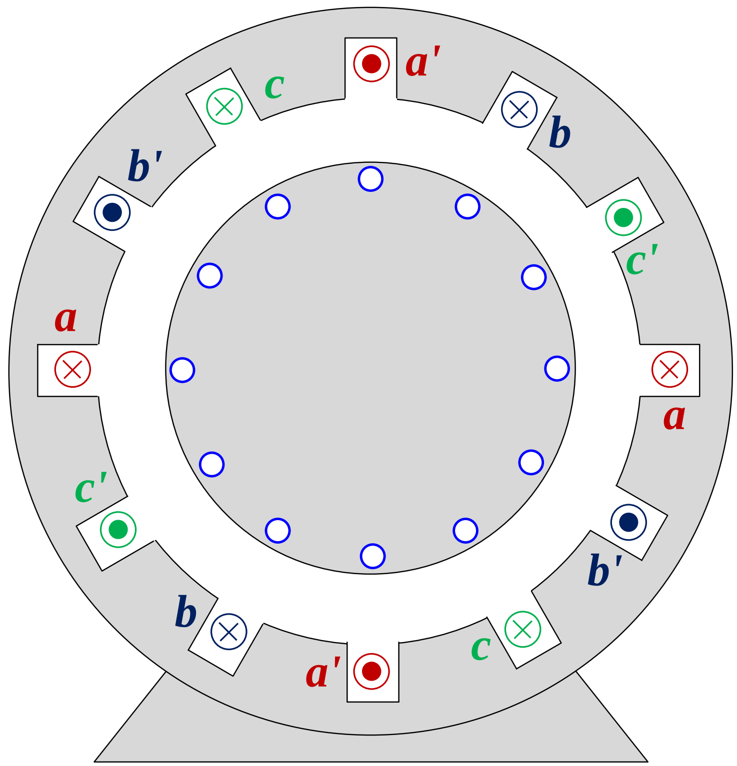
Slip is a dimensionless quantity.

How Does an Induction Motor Work?

1. Connecting the 3-phase stator windings to a 3-phase AC source flows the **current in the stator windings**.
2. The stator current causes a **rotating magnetic field** with synchronous speed.
3. The rotating magnetic field, **induces a voltage** in the rotor bars.
4. Since the rotor bars are short-circuited by end-rings, a **current flows in the rotor bars**.
5. The rotor bar current produces **another rotating magnetic field** which rotates with synchronous speed in the same direction as the stator magnetic field.
6. Electromagnetic **torque** is developed due to the interaction between two magnetic fields. $T_{em} = kB_r \times B_s$
7. The developed torque can **rotate** the rotor.

What is Meant by Asynchronous?

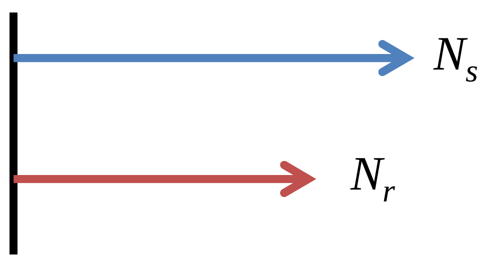
- The induced voltage in rotor bars is due to the **stator rotating magnetic field**.
- Therefore if the rotor rotates with **synchronous** speed, **no voltage** is induced in the rotor bars and no torque can be developed.
- Hence, to develop torque, the rotor speed should be different from the synchronous speed.
- It is because induction motors are often called **asynchronous** motors.

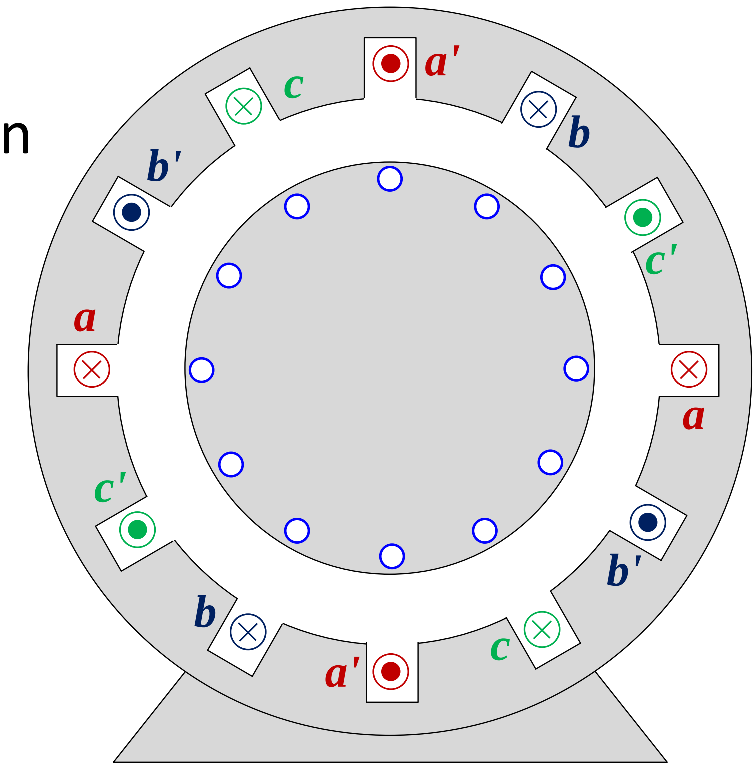


Operating Modes of Induction Machines

1. Motoring Mode

- The stator windings are connected to a 3-phase ac source.
- The mechanical energy is delivered on the motor shaft.





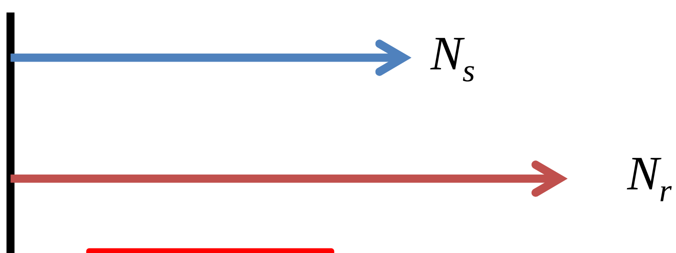
$$0 \le N_r \le N_s$$

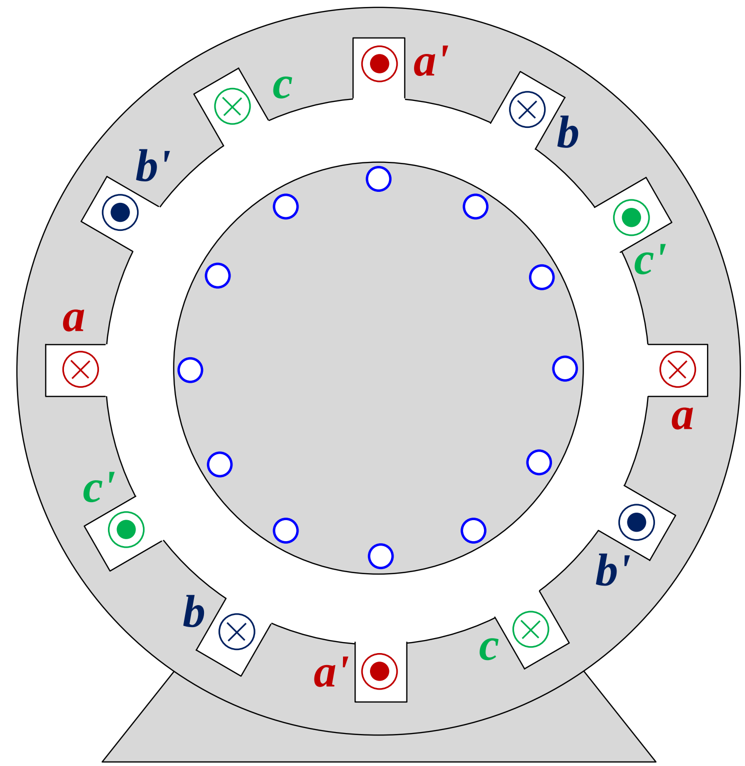
$$1 \ge S \ge 0$$

Operating Modes of Induction Machines

2. Generating Mode

- The stator windings are connected to a 3-phase ac source.
- The machine is in motoring mode.
- If by using a mechanical mover the rotor speed is increased to above synchronous speed, the machine will be a generator.



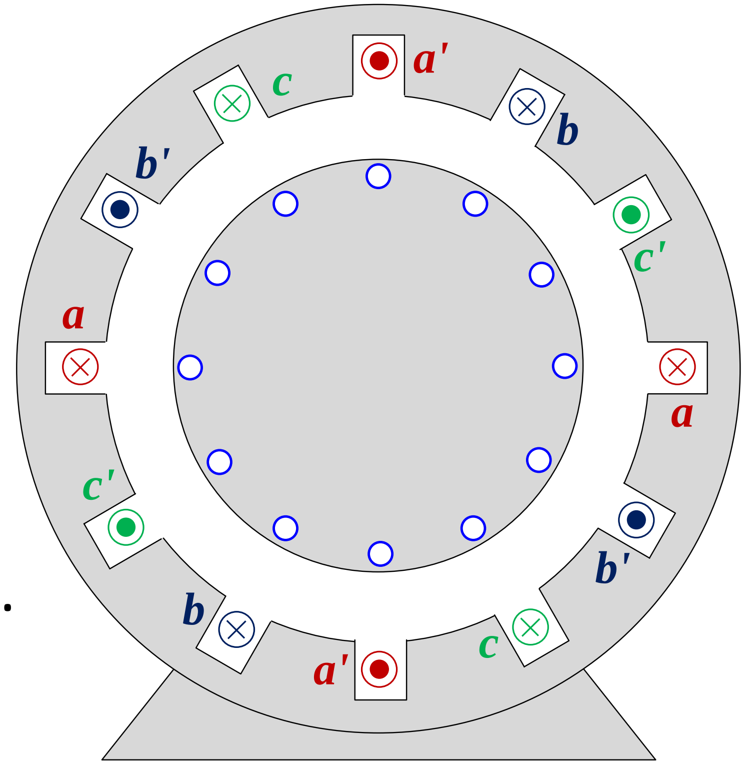


$$N_r > N_s$$

$$S < 0$$

Operating Modes of Induction Machines

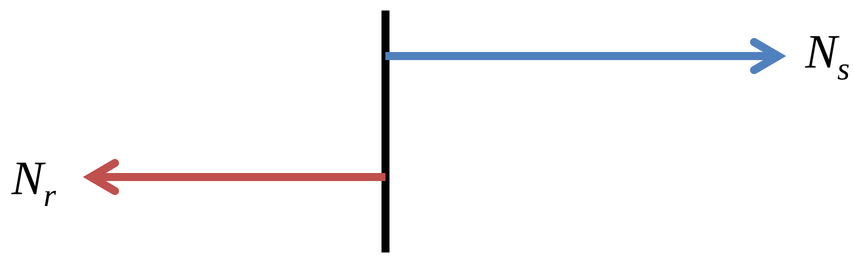
3. Plugging or Braking Mode



- If during the motoring mode the sequence of the applied voltage is changed,
- then the rotating magnetic field will change the direction,
- due to rotor inertia the magnetic field speed is in opposite of the rotor speed.
- The rotor will change the direction of rotation if it is not disconnected from the source.

$$N_r < 0$$

$$S > 1$$



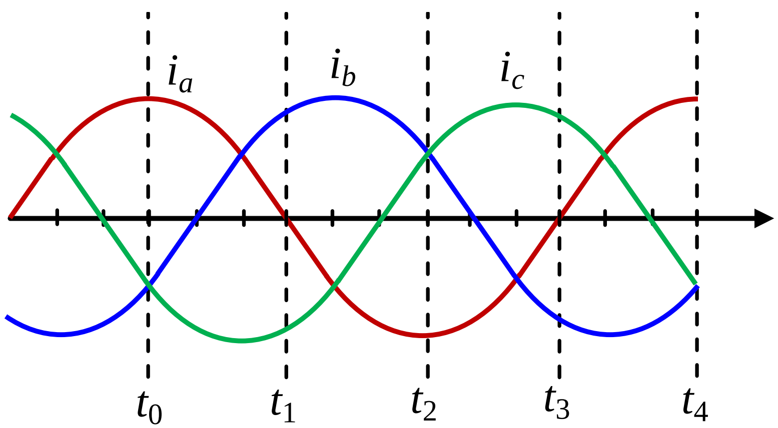
Rotating Magnetic Field

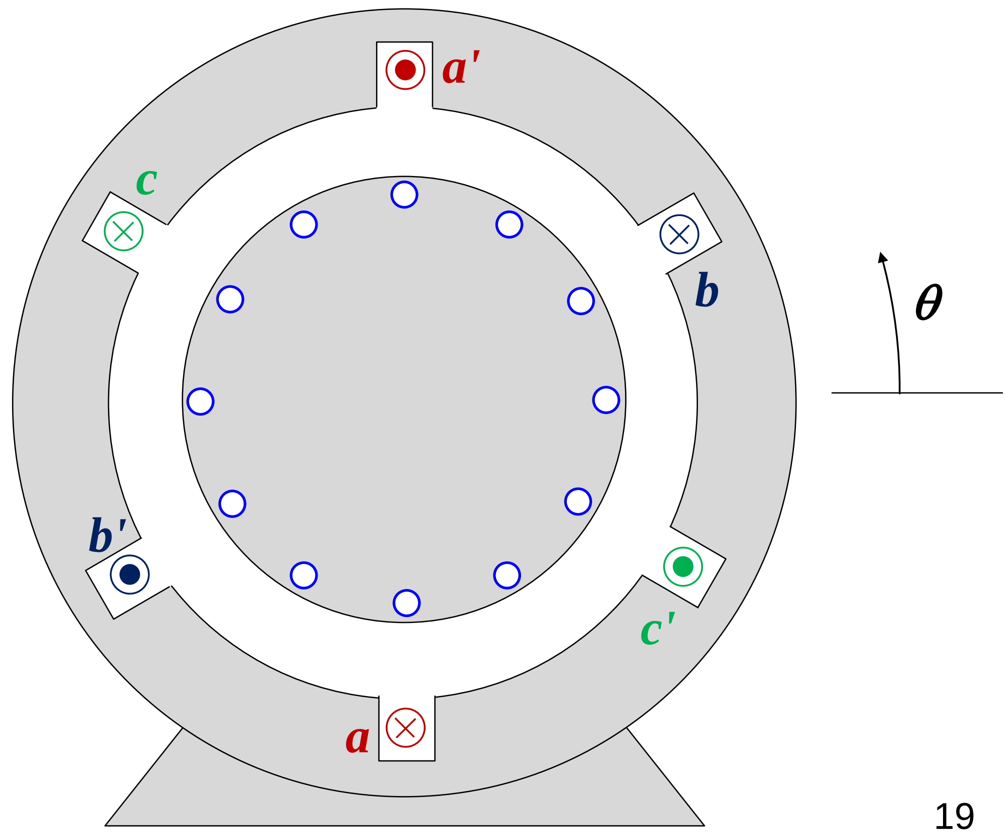
Consider the following **3-phase AC** machine with **2 poles** and **concentrated winding**. The 3-phase AC currents can be expressed as:

$$i_a(t) = I_m \cos \omega t$$

$$i_b(t) = I_m \cos(\omega t - 120^\circ)$$

$$i_c(t) = I_m \cos(\omega t + 120^\circ)$$



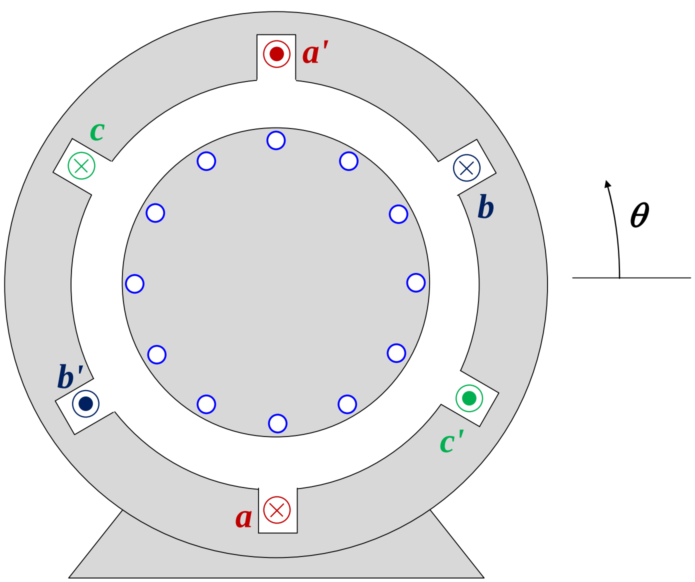


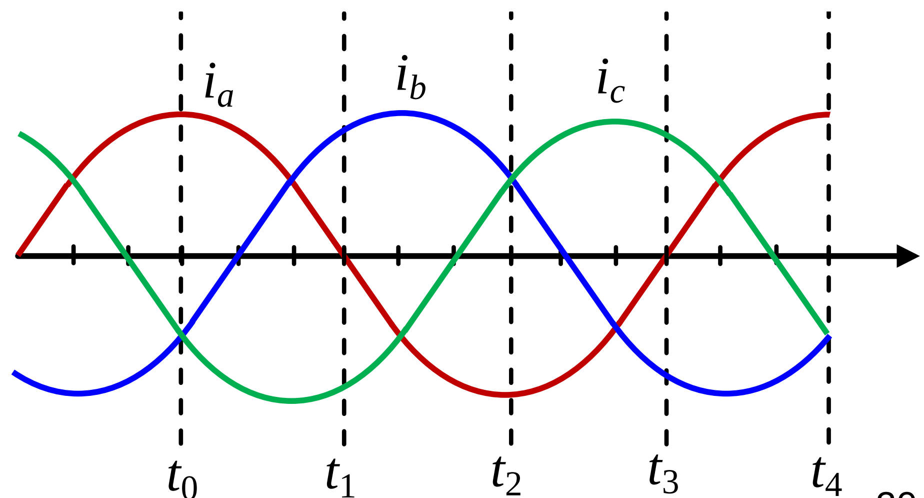
Rotating Magnetic Field

The rotating magnetomotive force (MMF) can be expressed as

$$\vec{F}(\theta,t) = \vec{F}_a(\theta,t) + \vec{F}_b(\theta,t) + \vec{F}_c(\theta,t)$$

$$\vec{F}(\theta,t) = Ni_a(t)\cos\theta + Ni_b(t)\cos(\theta - 120^\circ) + Ni_c(t)\cos(\theta + 120^\circ)$$





Rotating Magnetic Field

The rotating magnetomotive force (MMF) can be expressed as

$$\vec{F}(\theta,t) = Ni_a(t)\cos\theta + Ni_b(t)\cos(\theta-120^\circ) + Ni_c(t)\cos(\theta+120^\circ)$$

$$\begin{aligned}\vec{F}(\theta,t) = {} & NI_m \cos\omega t \cos\theta \\ & + NI_m \cos(\omega t - 120^\circ)\cos(\theta-120^\circ) \\ & + NI_m \cos(\omega t + 120^\circ)\cos(\theta+120^\circ)\end{aligned}$$

$$\vec{F}(\theta,t) = \tfrac{3}{2} NI_m \cos(\omega t - \theta)$$

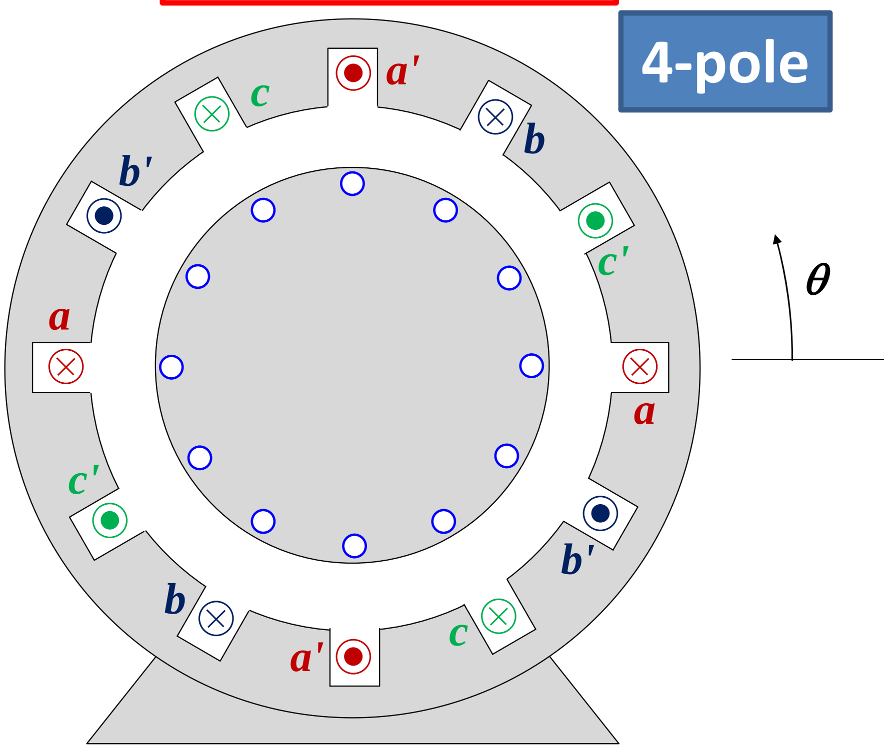
$$\cos a \cos b = \tfrac{1}{2}\cos(a-b) + \tfrac{1}{2}\cos(a+b)$$

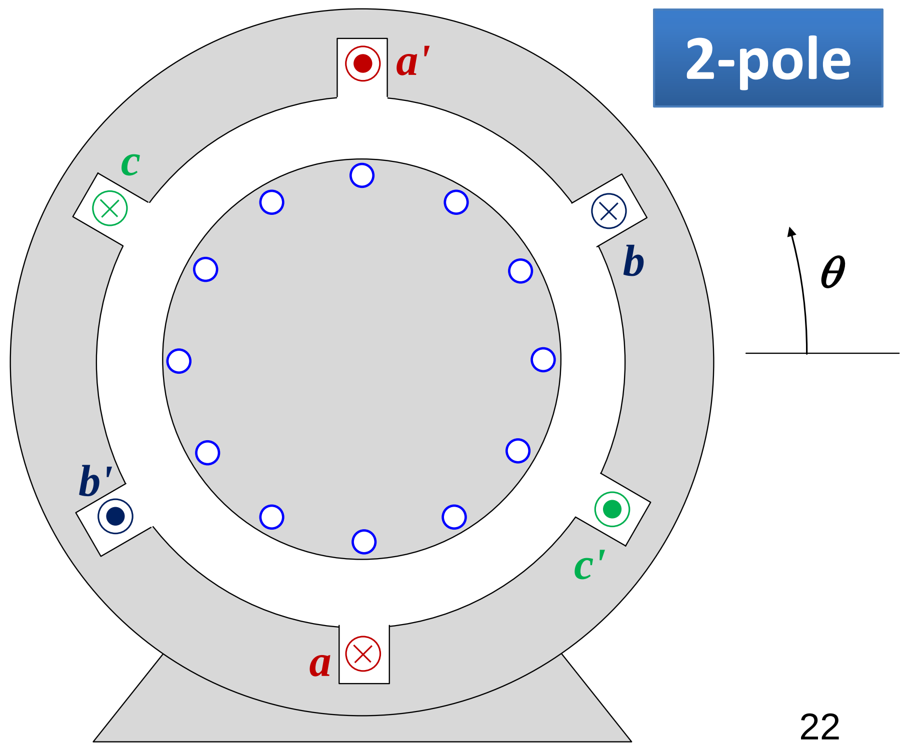
Electrical & Mechanical Angles

Consider an AC machine with P poles.

$$\theta_{elec} = \frac{P}{2}\theta_{mech}$$

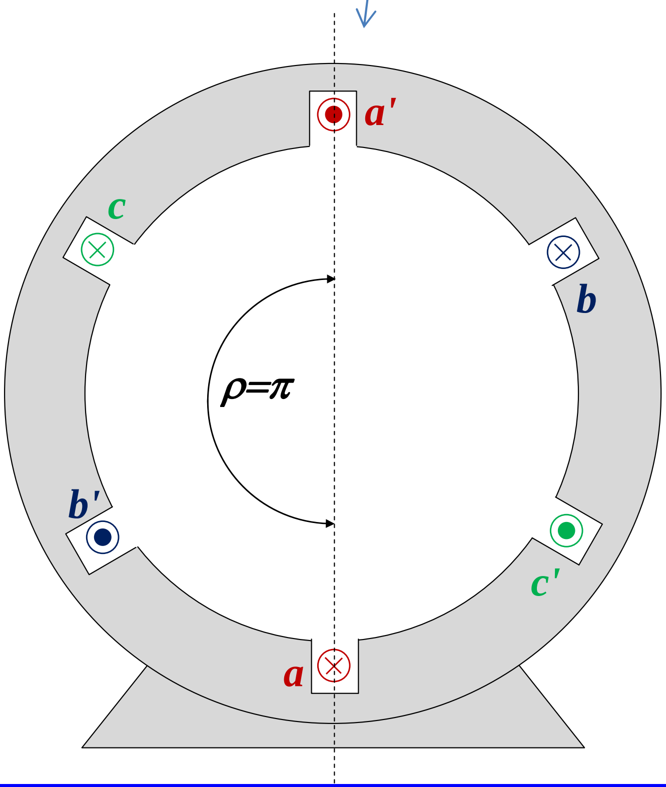
$$\omega_{elec} = \frac{P}{2}\omega_{mech}$$

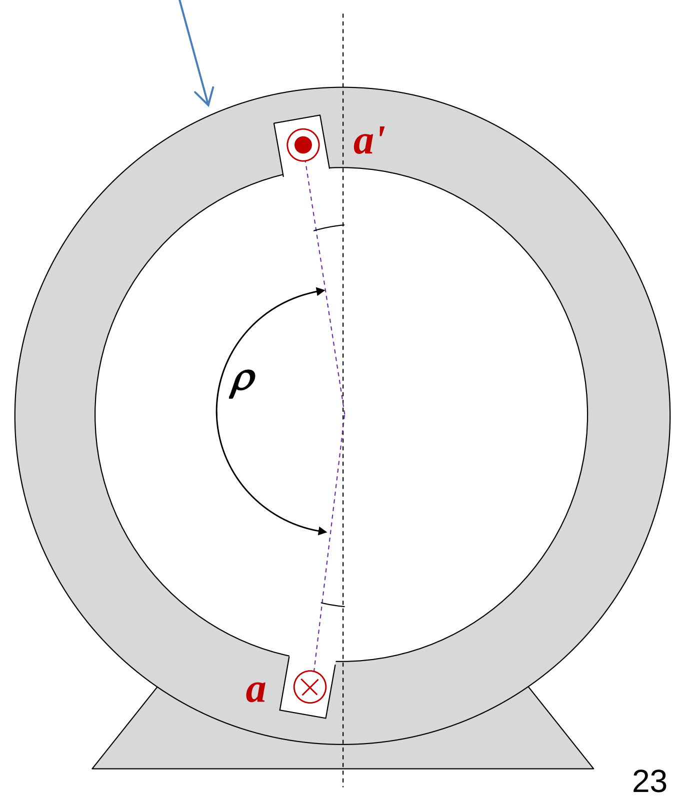




Coil Pitch

- **Coil pitch**: is the angle between two sides of one armature coil in electrical angle. If the coil pitch is 180 electrical degrees, the coil is a **full-pitch coil**; otherwise it is called **short**- or **chorded-pitch coil**.





Pitch Factor

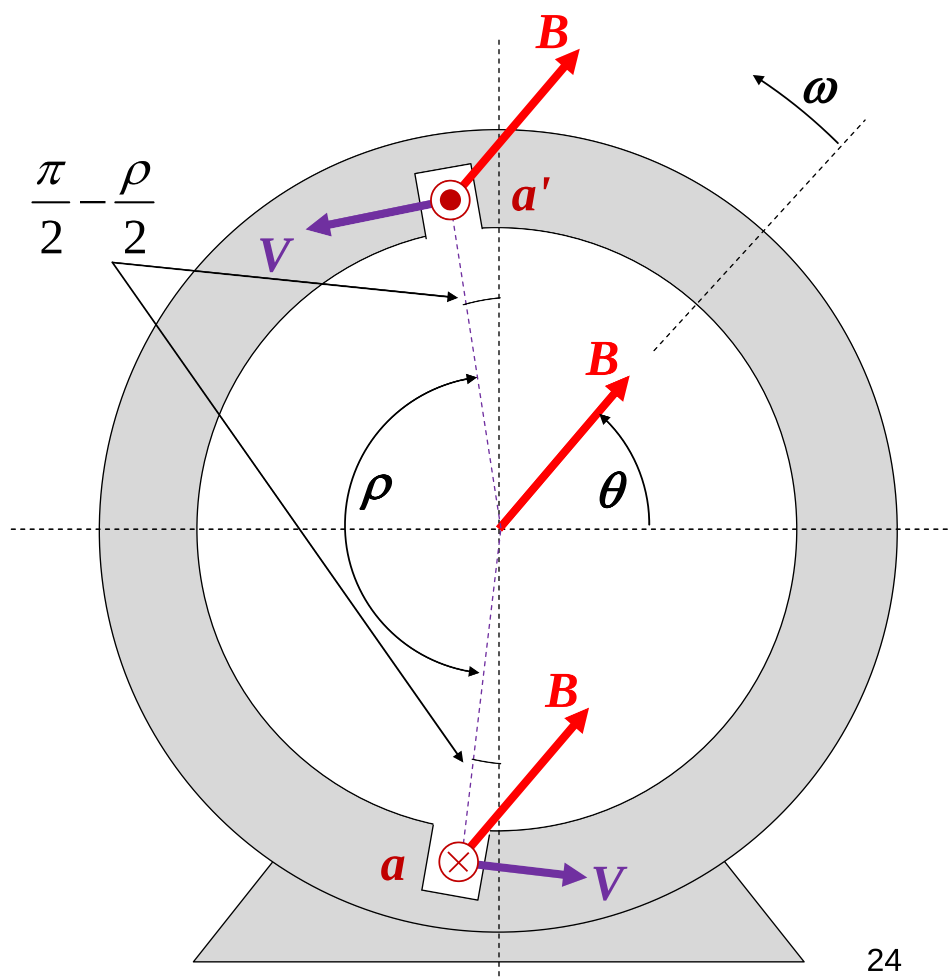
- Assume the axial length of the stator is l and the coil pitch is ρ as shown in the figure

$$e = (\vec{V} \times \vec{B}) \cdot \vec{l}$$

$$e = VBl \sin(\vec{V}, \vec{B})$$

$$\begin{cases} e_a = VBl \sin\left(\theta + \frac{\pi}{2} - \frac{\rho}{2}\right) \\ e_{a'} = VBl \sin\left(\pi + \frac{\pi}{2} - \frac{\rho}{2} - \theta\right) \end{cases}$$

$$\begin{cases} e_a = VBl \cos\left(\theta - \frac{\rho}{2}\right) \\ e_{a'} = -VBl \cos\left(\theta + \frac{\rho}{2}\right) \end{cases}$$

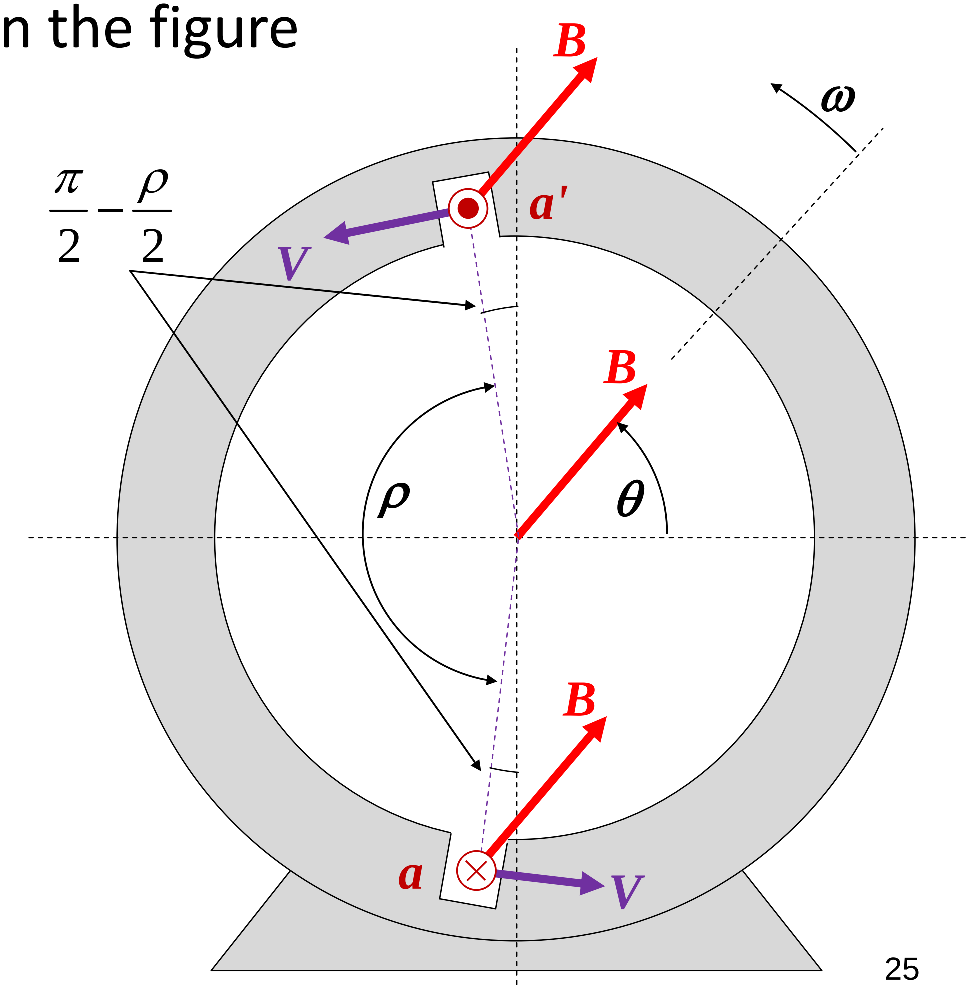


Pitch Factor

- Assume the axial length of the stator is l and the **coil pitch** is ρ in **electrical angle** as shown in the figure

$$\begin{cases} e_a = VBl\cos\left(\theta - \frac{\rho}{2}\right) \\ e_{a'} = -VBl\cos\left(\theta + \frac{\rho}{2}\right) \end{cases}$$

$$\left|e_{aa'}\right| = 2VBl\sin\theta\sin\left(\frac{\rho}{2}\right)$$



Pitch Factor

$$\left|e_{aa'}\right|_{Full-pitch} = 2VBl\sin\theta$$

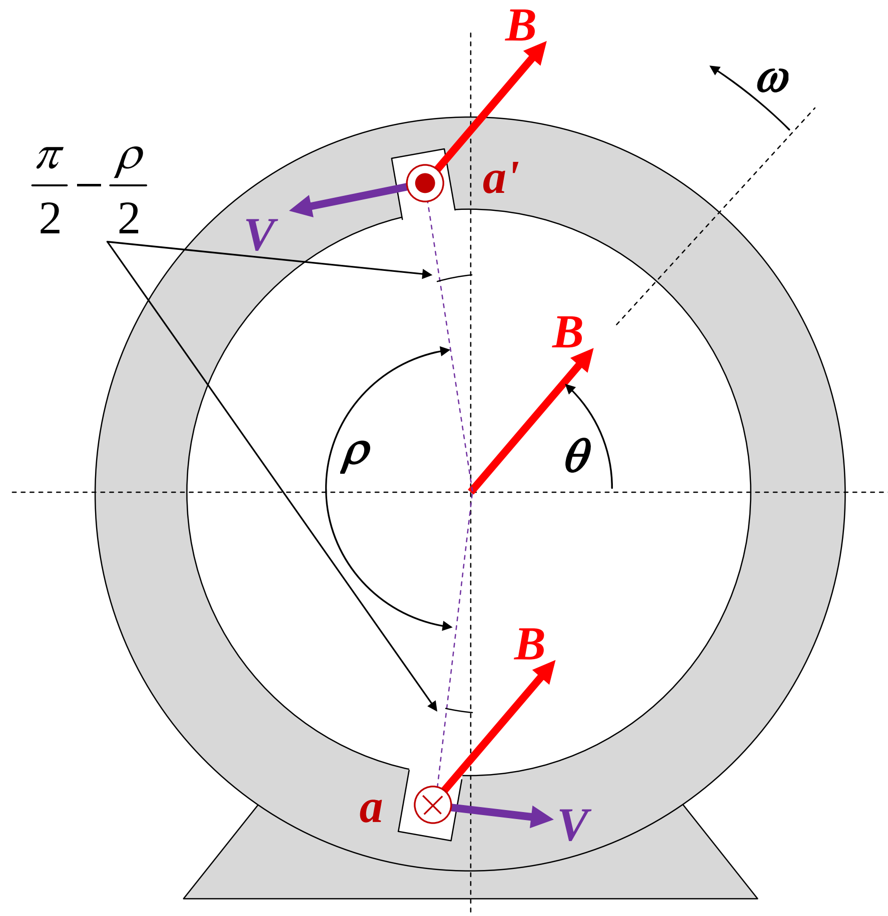
$$\left|e_{aa'}\right|_{Short-pitch} = 2VBl\sin\theta\sin\left(\tfrac{\rho}{2}\right)$$

The pitch factor is defined as

$$k_p = \frac{\left|e_{aa'}\right|_{\text{Short-pitch}}}{\left|e_{aa'}\right|_{\text{Full-pitch}}}$$

Therefore the pitch factor is

$$k_p = \sin\left(\tfrac{\rho}{2}\right)$$



Pitch Factor

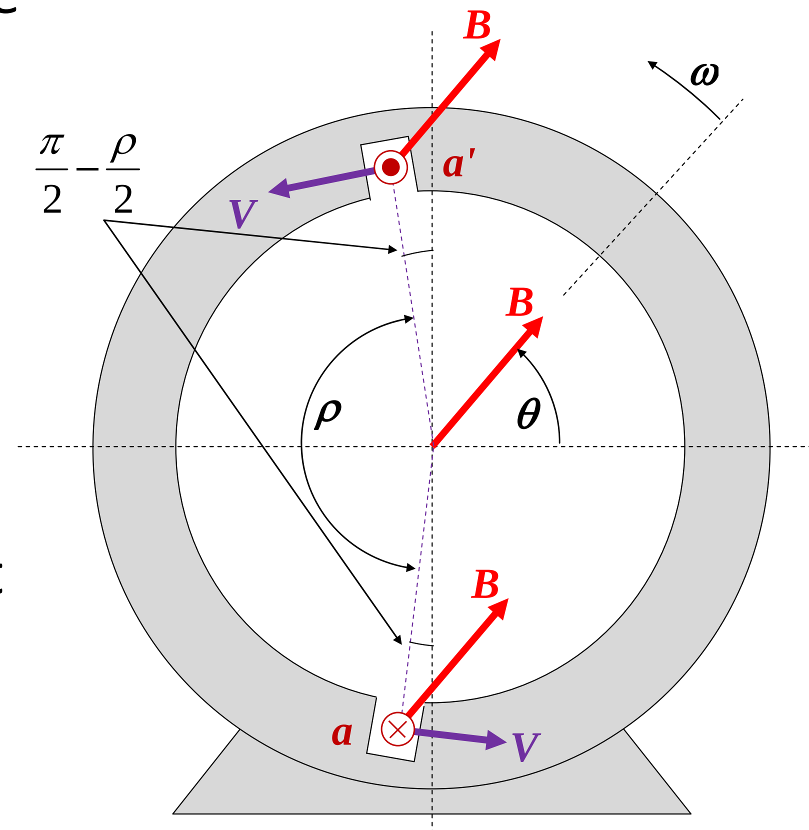
- Since the induced voltage may not be ideal sinusoidal, the pitch factor is expressed for all harmonics

$$k_{pn} = \sin\left(\frac{n\rho}{2}\right)$$

- **Why is short-pitch coil used?**

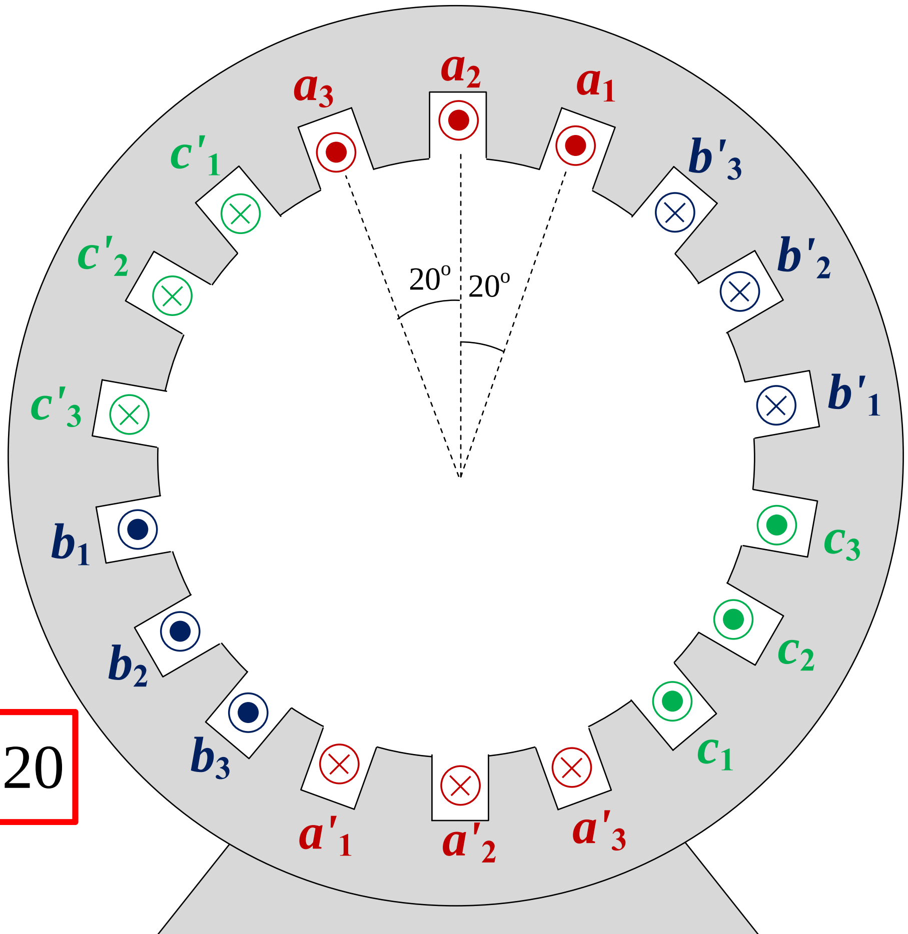
- Although short-pitch coil decreases the induced voltage (by about 3%), it **decreases** the **disturbing harmonics** significantly (by about 70 to 80%).

- Disturbing harmonics in electric machines are 5 and 7;



Distributed Windings

- To make the **MMF more sinusoidal**, distributed windings are used.
- Consider the following 2-pole single-layer distributed winding AC machine:
- The induced voltage in phase *a* is:



$$\vec{E}_{aa'} = \vec{E}_{aa'1} + \vec{E}_{aa'2} + \vec{E}_{aa'3}$$

$$\vec{E}_{aa'} = E_{rms}\angle -20 + E_{rms}\angle 0 + E_{rms}\angle 20$$

$$\vec{E}_{aa'} = 2.88 E_{rms}\angle 0$$

Distribution Factor

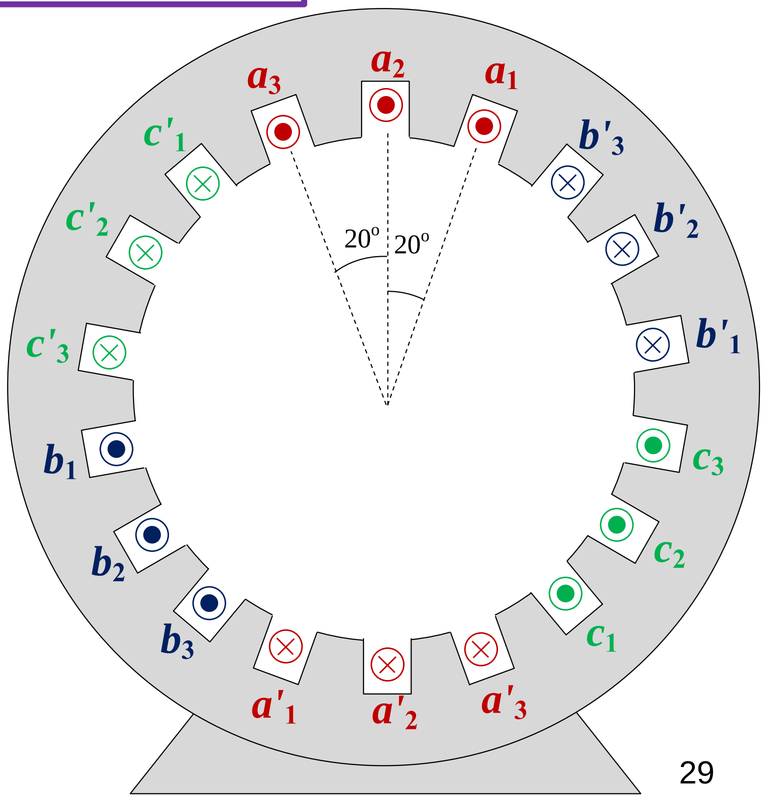
- Distribution factor is defined as:

$$k_d = \frac{\text{Induced voltage for distributed winding}}{\text{Induced voltage for concentrated winding}}$$

- In previous example

$$k_d = \frac{2.88E_{rms}}{3E_{rms}}$$

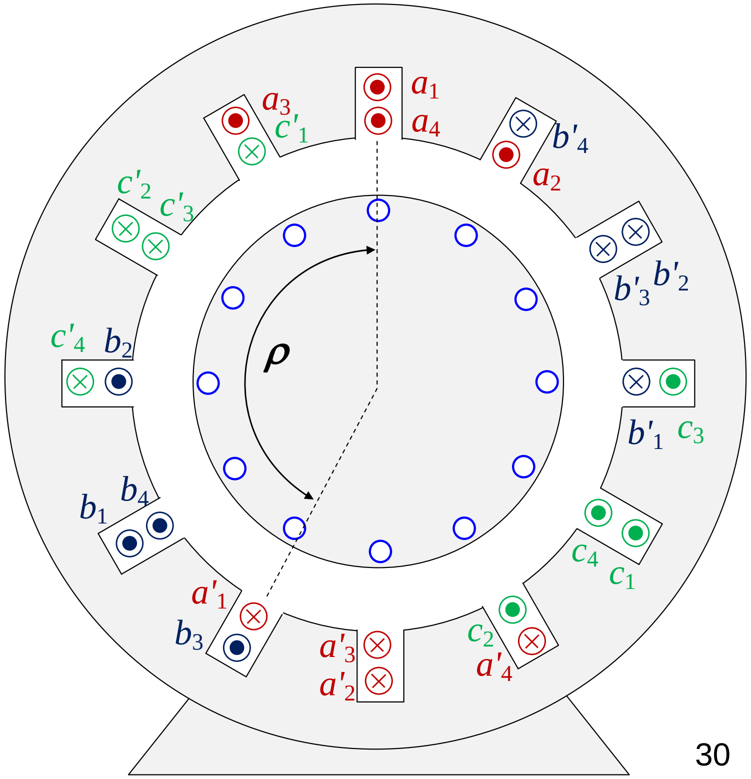
$$k_d = 0.96$$



Multi-layer Winding

- Most of AC machines, especially large machines, have **2-layer** winding.
- 2-layer winding has the following **advantages** compared to single-layer winding:

  1. Simpler winding
  2. Simpler connection
  3. Higher mechanical strength
  4. Optimal use of slots
  5. Lower cost



Distribution Factor

- Distribution factor is defined as:

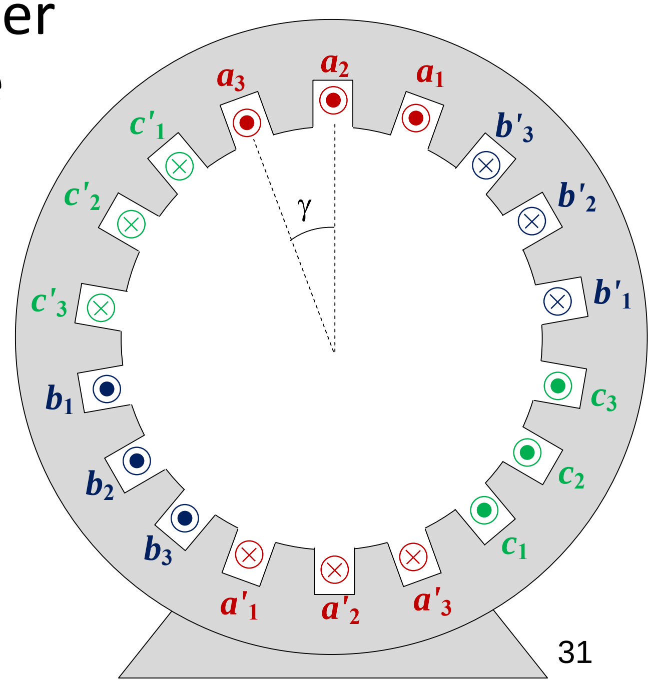
$$k_d = \frac{\text{Induced voltage for distributed winding}}{\text{Induced voltage for concentrated winding}}$$

- Assume the angle between two adjacent slots is γ in electrical angle and the number of slots per pole per phase is m, therefore the distribution factor is written as

$$k_d = \frac{\sin\frac{m\gamma}{2}}{m\sin\frac{\gamma}{2}}$$

- In n-th harmonic it is

$$k_{dn} = \frac{\sin\frac{mn\gamma}{2}}{m\sin\frac{n\gamma}{2}}$$



Winding Factor

- Winding factor is defined as

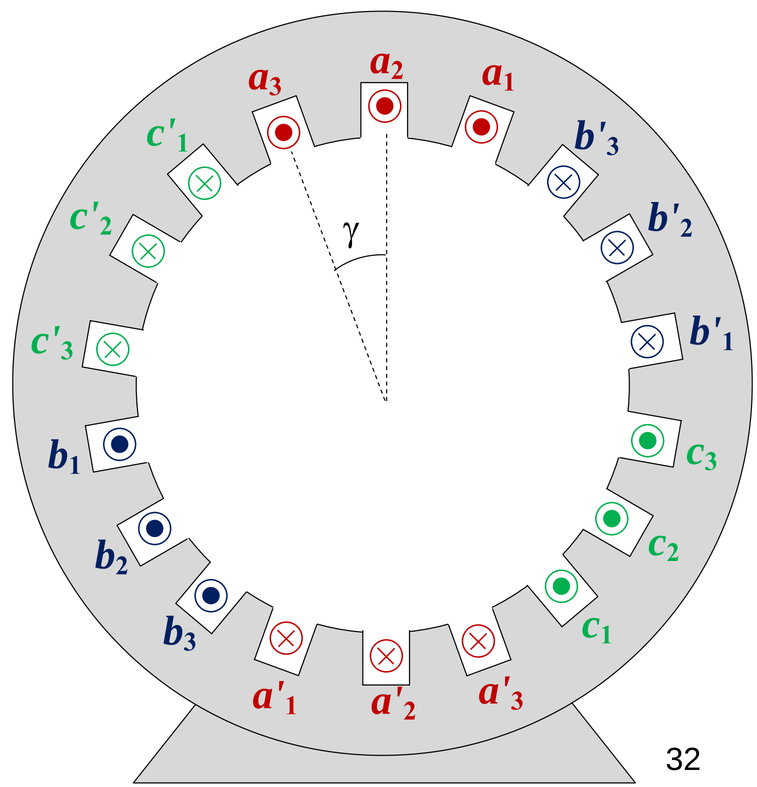
$$k_w = k_p k_d$$

$$k_p = \sin\left(\frac{\rho}{2}\right)$$

$$k_d = \frac{\sin \frac{m\gamma}{2}}{m \sin \frac{\gamma}{2}}$$

$$k_{pn} = \sin\left(\frac{n\rho}{2}\right)$$

$$k_{dn} = \frac{\sin \frac{m n \gamma}{2}}{m \sin \frac{n \gamma}{2}}$$

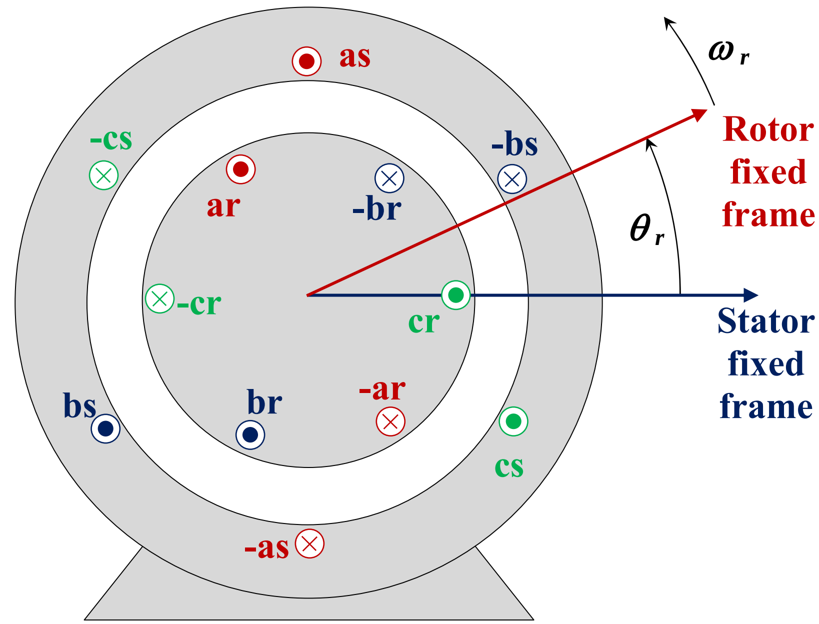


Induction Machine Modelling in *abc* System

1. Voltage equations

- **Stator voltage** equations in the **stator** fixed frame:

$$v_{as} = r_s i_{as} + d\lambda_{as}/dt$$
$$v_{bs} = r_s i_{bs} + d\lambda_{bs}/dt$$
$$v_{cs} = r_s i_{cs} + d\lambda_{cs}/dt$$



- **Rotor voltage** equations in the **rotor** fixed frame:

$$v_{ar} = r_r i_{ar} + d\lambda_{ar}/dt$$
$$v_{br} = r_r i_{br} + d\lambda_{br}/dt$$
$$v_{cr} = r_r i_{cr} + d\lambda_{cr}/dt$$

where r_s is the per-phase stator resistance and r_r is the per-phase rotor resistance.

Induction Machine Modelling in *abc* System

2. Flux linkage equations

- In matrix form, the flux linkages of the stator and rotor windings in terms of the winding inductances and currents are expressed as

$$\begin{bmatrix} \boldsymbol{\lambda}_s^{abc} \\ \boldsymbol{\lambda}_r^{abc} \end{bmatrix} = \begin{bmatrix} \mathbf{L}_{ss}^{abc} & \mathbf{L}_{sr}^{abc} \\ \mathbf{L}_{rs}^{abc} & \mathbf{L}_{rr}^{abc} \end{bmatrix} \begin{bmatrix} \mathbf{i}_s^{abc} \\ \mathbf{i}_r^{abc} \end{bmatrix}$$

where

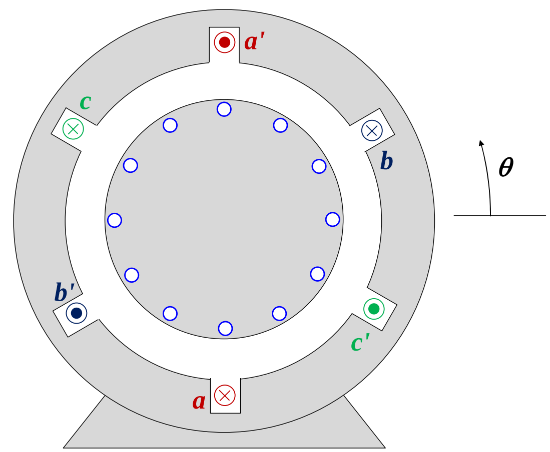
$$\boldsymbol{\lambda}_s^{abc} = \begin{bmatrix} \lambda_{as} \\ \lambda_{bs} \\ \lambda_{cs} \end{bmatrix} \quad \boldsymbol{\lambda}_r^{abc} = \begin{bmatrix} \lambda_{ar} \\ \lambda_{br} \\ \lambda_{cr} \end{bmatrix} \quad \mathbf{i}_s^{abc} = \begin{bmatrix} i_{as} \\ i_{bs} \\ i_{cs} \end{bmatrix} \quad \mathbf{i}_r^{abc} = \begin{bmatrix} i_{ar} \\ i_{br} \\ i_{cr} \end{bmatrix}$$

Induction Machine Modelling in *abc* System

2. Flux linkage equations

- The **stator-to-stator inductance** matrix is expressed as:

$$\mathbf{L}_{ss}^{abc} = \begin{bmatrix} L_{ls} + L_{ss} & L_{sm} & L_{sm} \\ L_{sm} & L_{ls} + L_{ss} & L_{sm} \\ L_{sm} & L_{sm} & L_{ls} + L_{ss} \end{bmatrix}$$



where

L_{ls} is the per-phase **stator** winding **leakage** inductance

L_{ss} is the per-phase **stator** winding **self-inductance**

L_{sm} is the **mutual-inductance** between **stator** windings.

Induction Machine Modelling in *abc* System

2. Flux linkage equations

- The **rotor-to-rotor inductance** matrix is expressed as:

$$\mathbf{L}_{rr}^{abc} = \begin{bmatrix} L_{lr} + L_{rr} & L_{rm} & L_{rm} \\ L_{rm} & L_{lr} + L_{rr} & L_{rm} \\ L_{rm} & L_{rm} & L_{lr} + L_{rr} \end{bmatrix}$$

where

L_{lr} is the per-phase **rotor** winding **leakage** inductance

L_{rr} is the per-phase **rotor** winding **self-inductance**

L_{rm} is the **mutual-inductance** between **rotor** windings.

Induction Machine Modelling in *abc* System

2. Flux linkage equations

- The stator-to-rotor mutual inductance matrix (which depends on the rotor angle) is expressed as:

$$\mathbf{L}_{sr}^{abc} = \left[\mathbf{L}_{rs}^{abc}\right]^T = L_{sr}\begin{bmatrix} \cos\theta_r & \cos(\theta_r + 2\pi/3) & \cos(\theta_r - 2\pi/3) \\ \cos(\theta_r - 2\pi/3) & \cos\theta_r & \cos(\theta_r + 2\pi/3) \\ \cos(\theta_r + 2\pi/3) & \cos(\theta_r - 2\pi/3) & \cos\theta_r \end{bmatrix}$$

where

L_{sr} is the **peak** value of the stator-to-rotor mutual inductance

θ_r is the rotor angle w.r.t. the stator fixed frame

$$\theta_r = \omega_r t + \theta_{r0}$$

Induction Machine Modelling in *abc* System

2. Flux linkage equations

- If the reluctive drops in iron are neglected, the inductances can be expressed in terms of the air-gap permeance (P_g), the rotor winding turns (N_r) and the stator winding turns (N_s):

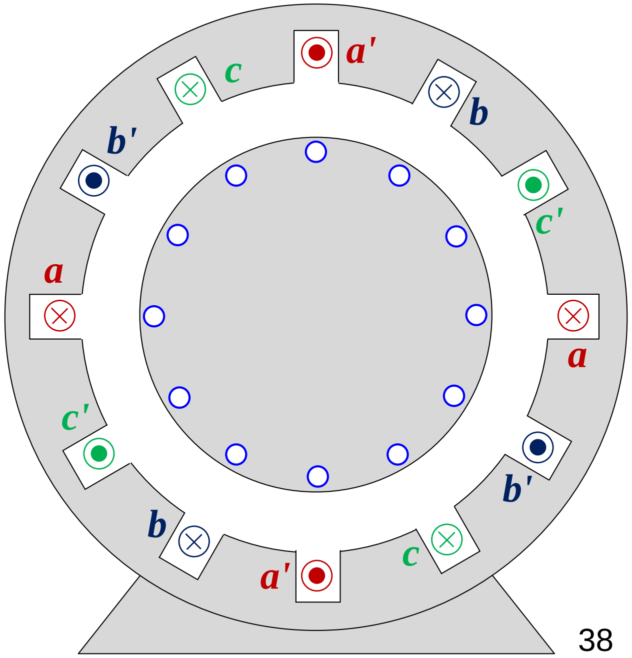
$$L_{ss} = N_s^2 P_g$$

$$L_{rr} = N_r^2 P_g$$

$$L_{sm} = N_s^2 P_g \cos(2\pi/3)$$

$$L_{rm} = N_r^2 P_g \cos(2\pi/3)$$

$$L_{sr} = N_s N_r P_g$$



Induction Machine Modelling in *abc* System

- Therefore the machine is described by **six first-order** differential equations

$$\begin{bmatrix} v_{as} \\ v_{bs} \\ v_{cs} \\ v_{ar} \\ v_{br} \\ v_{cr} \end{bmatrix} = \begin{bmatrix} r_s + pL_s & pL_{sm} & pL_{sm} & pL_{sr}\cos\theta_r & pL_{sr}\cos\theta_1 & pL_{sr}\cos\theta_2 \\ pL_{sm} & r_s + pL_s & pL_{sm} & pL_{sr}\cos\theta_2 & pL_{sr}\cos\theta_r & pL_{sr}\cos\theta_1 \\ pL_{sm} & pL_{sm} & r_s + pL_s & pL_{sr}\cos\theta_1 & pL_{sr}\cos\theta_2 & pL_{sr}\cos\theta_r \\ pL_{sr}\cos\theta_r & pL_{sr}\cos\theta_2 & pL_{sr}\cos\theta_1 & r_r + pL_r & pL_{rm} & pL_{rm} \\ pL_{sr}\cos\theta_1 & pL_{sr}\cos\theta_r & pL_{sr}\cos\theta_2 & pL_{rm} & r_r + pL_r & pL_{rm} \\ pL_{sr}\cos\theta_2 & pL_{sr}\cos\theta_1 & pL_{sr}\cos\theta_r & pL_{rm} & pL_{rm} & r_r + pL_r \end{bmatrix} \begin{bmatrix} i_{as} \\ i_{bs} \\ i_{cs} \\ i_{ar} \\ i_{br} \\ i_{cr} \end{bmatrix}$$

where $\theta_1 = \theta_r + 2\pi/3$ $\theta_2 = \theta_r - 2\pi/3$ $p = d/dt$

$L_s = L_{ls} + L_{ss}$ $L_r = L_{lr} + L_{rr}$

Induction Machine Modelling in *abc* System

- It can be written in the compact matrix form as follows

$$\begin{bmatrix} \mathbf{v}_s^{abc} \\ \mathbf{v}_r^{abc} \end{bmatrix} = \begin{bmatrix} \mathbf{r}_s^{abc} & 0 \\ 0 & \mathbf{r}_r^{abc} \end{bmatrix} \begin{bmatrix} \mathbf{i}_s^{abc} \\ \mathbf{i}_r^{abc} \end{bmatrix} + \frac{d}{dt} \begin{bmatrix} \mathbf{L}_{ss}^{abc} & \mathbf{L}_{sr}^{abc} \\ \mathbf{L}_{rs}^{abc} & \mathbf{L}_{rr}^{abc} \end{bmatrix} \begin{bmatrix} \mathbf{i}_s^{abc} \\ \mathbf{i}_r^{abc} \end{bmatrix}$$

where

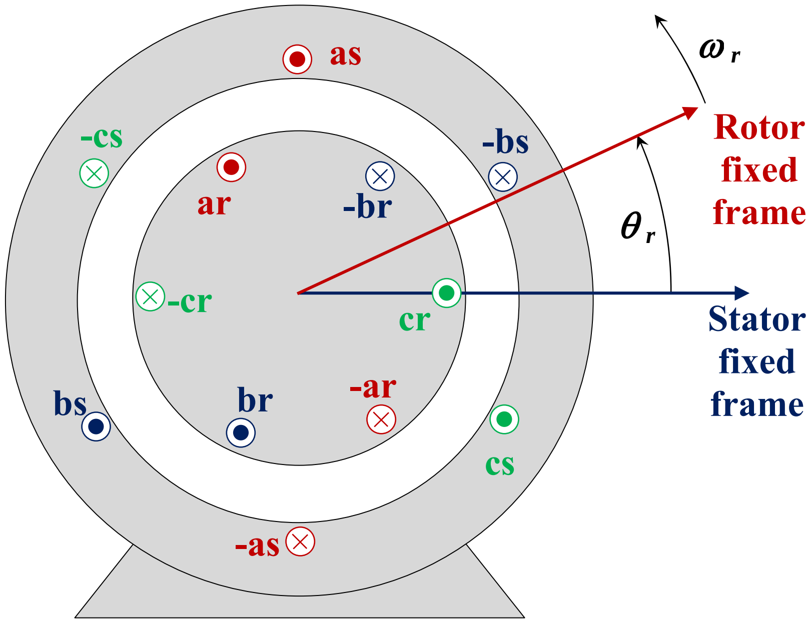
$$\mathbf{r}_s^{abc} = \begin{bmatrix} r_s & 0 & 0 \\ 0 & r_s & 0 \\ 0 & 0 & r_s \end{bmatrix} \qquad \mathbf{v}_s^{abc} = \begin{bmatrix} v_{as} \\ v_{bs} \\ v_{cs} \end{bmatrix} \qquad \mathbf{i}_s^{abc} = \begin{bmatrix} i_{as} \\ i_{bs} \\ i_{cs} \end{bmatrix}$$

$$\mathbf{r}_r^{abc} = \begin{bmatrix} r_r & 0 & 0 \\ 0 & r_r & 0 \\ 0 & 0 & r_r \end{bmatrix} \qquad \mathbf{v}_r^{abc} = \begin{bmatrix} v_{ar} \\ v_{br} \\ v_{cr} \end{bmatrix} \qquad \mathbf{i}_r^{abc} = \begin{bmatrix} i_{ar} \\ i_{br} \\ i_{cr} \end{bmatrix}$$

Induction Machine Modelling in *abc* System

$$\mathbf{L}_{ss}^{abc}=\begin{bmatrix} L_{ls}+L_{ss} & L_{sm} & L_{sm} \\ L_{sm} & L_{ls}+L_{ss} & L_{sm} \\ L_{sm} & L_{sm} & L_{ls}+L_{ss} \end{bmatrix}$$

$$\mathbf{L}_{rr}^{abc}=\begin{bmatrix} L_{lr}+L_{rr} & L_{rm} & L_{rm} \\ L_{rm} & L_{lr}+L_{rr} & L_{rm} \\ L_{rm} & L_{rm} & L_{lr}+L_{rr} \end{bmatrix}$$



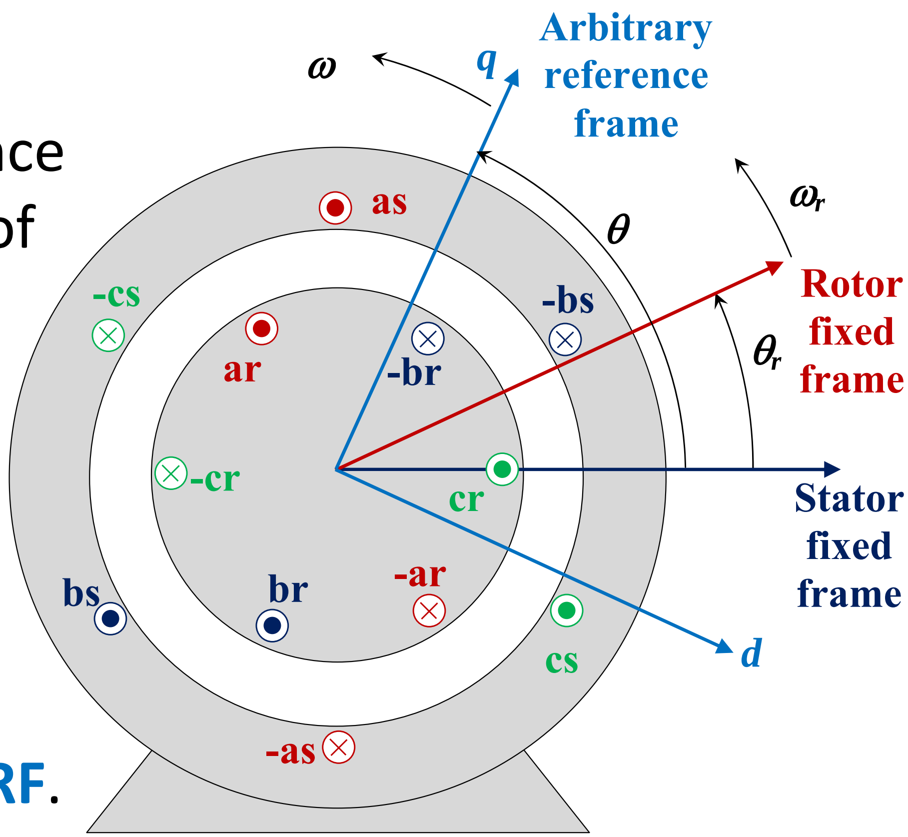
$$\mathbf{L}_{sr}^{abc}=\left[\mathbf{L}_{rs}^{abc}\right]^{T}=L_{sr}\begin{bmatrix} \cos\theta_r & \cos(\theta_r+2\pi/3) & \cos(\theta_r-2\pi/3) \\ \cos(\theta_r-2\pi/3) & \cos\theta_r & \cos(\theta_r+2\pi/3) \\ \cos(\theta_r+2\pi/3) & \cos(\theta_r-2\pi/3) & \cos\theta_r \end{bmatrix}$$

Induction Machine Modelling in *abc* System

- As mentioned before the machine model in abc system is represented by **six first-order** differential equations.
- These differential equations are **coupled** due to the mutual inductances.
- The stator-to-rotor coupling terms are **functions of rotor position**; thus when rotor rotates, they **vary with time**.
- Mathematical transformations like dq or $\alpha\beta$ can **facilitate** the computation by transferring the differential equations with time-varying inductances to DEs with constant inductances.

Induction Machine Modelling in Arbitrary $qd0$ RF

- The equations of IM are first derived in the **arbitrary** reference frame (RF) rotating at a speed of ω in the direction of the rotor rotation.
- Setting $\omega = 0$ yields the equations in the **stationary RF**.
- Setting $\omega = \omega_e$ yields the equations in the **synchronous RF**.



Induction Machine Modelling in Arbitrary *qd*0 RF

- The transformation is expressed as

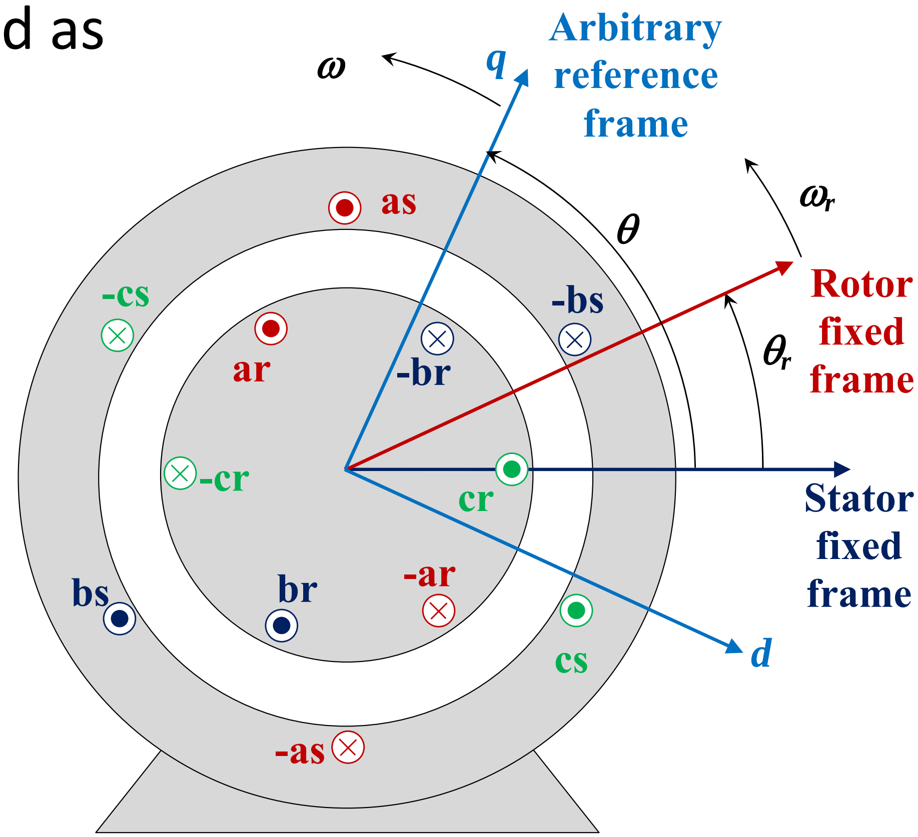
$$\left[\mathbf{f}_{qd0}\right] = \left[\mathbf{T}_{qd0}(\theta)\right]\left[\mathbf{f}_{abc}\right]$$

- The transformation angle is

$$\theta(t) = \int_0^t \omega(t)\,dt + \theta(0)$$

- The rotor angle is expressed as

$$\theta_r(t) = \int_0^t \omega_r(t)\,dt + \theta_r(0)$$



Induction Machine Modelling in Arbitrary $qd0$ RF

- The transformation and its inverse for **stator** equations

$$\left[\mathbf{f}_{qd0}\right]=\left[\mathbf{T}_{qd0}(\theta)\right]\ \left[\mathbf{f}_{abc}\right]$$

$$\left[\mathbf{f}_{abc}\right]=\left[\mathbf{T}_{qd0}(\theta)\right]^{-1}\left[\mathbf{f}_{qd0}\right]$$

$$\left[\mathbf{T}_{qd0}(\theta)\right]=\frac{2}{3}\begin{bmatrix}\cos\theta & \cos\left(\theta-2\pi/3\right) & \cos\left(\theta+2\pi/3\right)\\ \sin\theta & \sin\left(\theta-2\pi/3\right) & \sin\left(\theta+2\pi/3\right)\\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\end{bmatrix}$$

$$\left[\mathbf{T}_{qd0}(\theta)\right]^{-1}=\begin{bmatrix}\cos\theta & \sin\theta & 1\\ \cos\left(\theta-2\pi/3\right) & \sin\left(\theta-2\pi/3\right) & 1\\ \cos\left(\theta+2\pi/3\right) & \sin\left(\theta+2\pi/3\right) & 1\end{bmatrix}$$

Induction Machine Modelling in Arbitrary $qd0$ RF

- The transformation and its inverse for **rotor** equations

$$\left[\mathbf{f}_{qd0}\right]=\left[\mathbf{T}_{qd0}(\theta-\theta_r)\right]\left[\mathbf{f}_{abc}\right]$$

$$\left[\mathbf{f}_{abc}\right]=\left[\mathbf{T}_{qd0}(\theta-\theta_r)\right]^{-1}\left[\mathbf{f}_{qd0}\right]$$

$$\left[\mathbf{T}_{qd0}(\theta-\theta_r)\right]=\frac{2}{3}\begin{bmatrix}\cos(\theta-\theta_r) & \cos(\theta-\theta_r-2\pi/3) & \cos(\theta-\theta_r+2\pi/3)\\ \sin(\theta-\theta_r) & \sin(\theta-\theta_r-2\pi/3) & \sin(\theta-\theta_r+2\pi/3)\\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\end{bmatrix}$$

$$\left[\mathbf{T}_{qd0}(\theta-\theta_r)\right]^{-1}=\begin{bmatrix}\cos(\theta-\theta_r) & \sin(\theta-\theta_r) & 1\\ \cos(\theta-\theta_r-2\pi/3) & \sin(\theta-\theta_r-2\pi/3) & 1\\ \cos(\theta-\theta_r+2\pi/3) & \sin(\theta-\theta_r+2\pi/3) & 1\end{bmatrix}$$

Induction Machine Modelling in Arbitrary $qd0$ RF

qd0 voltage equations

Stator

- The **stator voltage** equations in abc system is expressed as:

$$\mathbf{v}_s^{abc} = \mathbf{r}_s^{abc}\mathbf{i}_s^{abc} + p\boldsymbol{\lambda}_s^{abc}$$

where $p = d/dt$

- Applying the transformation $\mathbf{T}_{qd0}(\theta)$ to voltage, current and flux yields

$$[\mathbf{f}_{abc}] = [\mathbf{T}_{qd0}(\theta)]^{-1}[\mathbf{f}_{qd0}]$$

$$[\mathbf{T}_{qd0}(\theta)]^{-1}\mathbf{v}_s^{qd0} = \mathbf{r}_s^{abc}[\mathbf{T}_{qd0}(\theta)]^{-1}\mathbf{i}_s^{qd0} + p\left([\mathbf{T}_{qd0}(\theta)]^{-1}\boldsymbol{\lambda}_s^{qd0}\right)$$

- Multiplying both sides by the transformation matrix yields

$$\mathbf{v}_s^{qd0} = [\mathbf{T}_{qd0}(\theta)]\mathbf{r}_s^{abc}[\mathbf{T}_{qd0}(\theta)]^{-1}\mathbf{i}_s^{qd0} + [\mathbf{T}_{qd0}(\theta)]p\left([\mathbf{T}_{qd0}(\theta)]^{-1}\boldsymbol{\lambda}_s^{qd0}\right)$$

Induction Machine Modelling in Arbitrary $qd0$ RF

qd0 voltage equations

Stator

$$\mathbf{v}_s^{qd0} = \left[\mathbf{T}_{qd0}(\theta)\right]\mathbf{r}_s^{abc}\left[\mathbf{T}_{qd0}(\theta)\right]^{-1}\mathbf{i}_s^{qd0} + \left[\mathbf{T}_{qd0}(\theta)\right]p\left(\left[\mathbf{T}_{qd0}(\theta)\right]^{-1}\boldsymbol{\lambda}_s^{qd0}\right)$$

- Substituting the following relations in the above expression

$$\mathbf{r}_s^{abc} = \mathbf{r}_s^{qd0} = r_s\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\left[\mathbf{T}_{qd0}(\theta)\right]p\left(\left[\mathbf{T}_{qd0}(\theta)\right]^{-1}\boldsymbol{\lambda}_s^{qd0}\right) = \omega\begin{bmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\boldsymbol{\lambda}_s^{qd0} + p\boldsymbol{\lambda}_s^{qd0}$$

yields

$$\mathbf{v}_s^{qd0} = \mathbf{r}_s^{qd0}\mathbf{i}_s^{qd0} + \omega\begin{bmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\boldsymbol{\lambda}_s^{qd0} + p\,\boldsymbol{\lambda}_s^{qd0}$$

Induction Machine Modelling in Arbitrary $qd0$ RF

Stator $qd0$ voltage equations

qd0 voltage equations

Stator

$$\mathbf{v}_s^{qd0} = \mathbf{r}_s^{qd0}\mathbf{i}_s^{qd0} + \omega \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \boldsymbol{\lambda}_s^{qd0} + p\,\boldsymbol{\lambda}_s^{qd0}$$

Ohmic drop — Rotational EMF — Transformer EMF

$$v_{qs} = r_s\, i_{qs} + p\lambda_{qs} + \omega\lambda_{ds}$$

$$v_{ds} = r_s\, i_{ds} + p\lambda_{ds} - \omega\lambda_{qs}$$

$$v_{0s} = r_s\, i_{0s} + p\lambda_{0s}$$

Induction Machine Modelling in Arbitrary $qd0$ RF

qd0 voltage equations

Rotor

- The **rotor voltage** equations in *abc* system is expressed as:

$$\mathbf{v}_r^{abc} = \mathbf{r}_r^{abc}\mathbf{i}_r^{abc} + p\boldsymbol{\lambda}_r^{abc}$$

- Similarly applying the transformation $\mathbf{T}_{qd0}(\theta-\theta_r)$ to voltage, current and flux yields

$$\left[\mathbf{f}_{abc}\right] = \left[\mathbf{T}_{qd0}(\theta-\theta_r)\right]^{-1}\left[\mathbf{f}_{qd0}\right]$$

$$\mathbf{v}_r^{qd0} = \mathbf{r}_r^{qd0}\mathbf{i}_r^{qd0} + (\omega-\omega_r)\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\boldsymbol{\lambda}_r^{qd0} + p\,\boldsymbol{\lambda}_r^{qd0}$$

Induction Machine Modelling in Arbitrary *qd*0 RF

Rotor *qd0* voltage equations

qd0 voltage equations

Rotor

$$\mathbf{v}_r^{qd0} = \mathbf{r}_r^{qd0}\mathbf{i}_r^{qd0} + (\omega - \omega_r)\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\boldsymbol{\lambda}_r^{qd0} + p\,\boldsymbol{\lambda}_r^{qd0}$$

Ohmic drop — Rotational EMF — Transformer EMF

$$v_{qr} = r_r\, i_{qr} + p\lambda_{qr} + (\omega - \omega_r)\,\lambda_{dr}$$

$$v_{dr} = r_r\, i_{dr} + p\lambda_{dr} - (\omega - \omega_r)\,\lambda_{qr}$$

$$v_{0r} = r_r\, i_{0r} + p\lambda_{0r}$$

Induction Machine Modelling in Arbitrary $qd0$ RF

qd0 voltage equations

Stator & Rotor

$$\begin{bmatrix} v_{qs} \\ v_{ds} \\ v_{0s} \\ \hline v_{qr} \\ v_{dr} \\ v_{0r} \end{bmatrix} = \left[\begin{array}{ccc|ccc} r_s & 0 & 0 & 0 & 0 & 0 \\ 0 & r_s & 0 & 0 & 0 & 0 \\ 0 & 0 & r_s & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & r_r & 0 & 0 \\ 0 & 0 & 0 & 0 & r_r & 0 \\ 0 & 0 & 0 & 0 & 0 & r_r \end{array}\right] \begin{bmatrix} i_{qs} \\ i_{ds} \\ i_{0s} \\ \hline i_{qr} \\ i_{dr} \\ i_{0r} \end{bmatrix} + \frac{d}{dt} \begin{bmatrix} \lambda_{qs} \\ \lambda_{ds} \\ \lambda_{0s} \\ \hline \lambda_{qr} \\ \lambda_{dr} \\ \lambda_{0r} \end{bmatrix}$$

$$+ \left[\begin{array}{ccc|ccc} 0 & \omega & 0 & 0 & 0 & 0 \\ -\omega & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \omega-\omega_r & 0 \\ 0 & 0 & 0 & -(\omega-\omega_r) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} \lambda_{qs} \\ \lambda_{ds} \\ \lambda_{0s} \\ \hline \lambda_{qr} \\ \lambda_{dr} \\ \lambda_{0r} \end{bmatrix}$$

It is required to refer all rotor quantities to the stator side.

Induction Machine Modelling in Arbitrary $qd0$ RF

qd0 voltage equations referred to stator side

Stator & Rotor

$$\begin{bmatrix} v_{qs} \\ v_{ds} \\ v_{0s} \\ \hline v'_{qr} \\ v'_{dr} \\ v'_{0r} \end{bmatrix} = \left[\begin{array}{ccc|ccc} r_s & 0 & 0 & 0 & 0 & 0 \\ 0 & r_s & 0 & 0 & 0 & 0 \\ 0 & 0 & r_s & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & r'_r & 0 & 0 \\ 0 & 0 & 0 & 0 & r'_r & 0 \\ 0 & 0 & 0 & 0 & 0 & r'_r \end{array}\right] \begin{bmatrix} i_{qs} \\ i_{ds} \\ i_{0s} \\ \hline i'_{qr} \\ i'_{dr} \\ i'_{0r} \end{bmatrix} + \frac{d}{dt} \begin{bmatrix} \lambda_{qs} \\ \lambda_{ds} \\ \lambda_{0s} \\ \hline \lambda'_{qr} \\ \lambda'_{dr} \\ \lambda'_{0r} \end{bmatrix}$$

$$+ \left[\begin{array}{ccc|ccc} 0 & \omega & 0 & 0 & 0 & 0 \\ -\omega & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \omega-\omega_r & 0 \\ 0 & 0 & 0 & -(\omega-\omega_r) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} \lambda_{qs} \\ \lambda_{ds} \\ \lambda_{0s} \\ \hline \lambda'_{qr} \\ \lambda'_{dr} \\ \lambda'_{0r} \end{bmatrix}$$

$$v'_{qr} = \frac{N_s}{N_r} v_{qr} \qquad \lambda'_{qr} = \frac{N_s}{N_r} \lambda_{qr}$$

$$v'_{dr} = \frac{N_s}{N_r} v_{dr} \qquad \lambda'_{dr} = \frac{N_s}{N_r} \lambda_{dr}$$

$$v'_{0r} = \frac{N_s}{N_r} v_{0r} \qquad \lambda'_{0r} = \frac{N_s}{N_r} \lambda_{0r}$$

$$i'_{qr} = \frac{N_r}{N_s} i_{qr}$$

$$i'_{dr} = \frac{N_r}{N_s} i_{dr} \qquad r'_r = \left(\frac{N_s}{N_r}\right)^2 r_r$$

$$i'_{0r} = \frac{N_r}{N_s} i_{0r}$$

Induction Machine Modelling in Arbitrary $qd0$ RF

_qd0_ flux linkage relation

Stator

- The **stator flux linkage** equations in *abc* system is expressed as:

$$\boldsymbol{\lambda}_s^{abc} = \mathbf{L}_{ss}^{abc}\,\mathbf{i}_s^{abc} + \mathbf{L}_{sr}^{abc}\,\mathbf{i}_r^{abc}$$

- Applying the transformation $\mathbf{T}_{qd0}(\theta)$ and $\mathbf{T}_{qd0}(\theta-\theta_r)$ to the stator and rotor quantities yields

$$\left[\mathbf{T}_{qd0}(\theta)\right]^{-1}\boldsymbol{\lambda}_s^{qd0} = \mathbf{L}_{ss}^{abc}\left[\mathbf{T}_{qd0}(\theta)\right]^{-1}\mathbf{i}_s^{qd0} + \mathbf{L}_{sr}^{abc}\left[\mathbf{T}_{qd0}(\theta-\theta_r)\right]^{-1}\mathbf{i}_r^{qd0}$$

- Multiplying both sides by $\mathbf{T}_{qd0}(\theta)$ yields

$$\boldsymbol{\lambda}_s^{qd0} = \left[\mathbf{T}_{qd0}(\theta)\right]\mathbf{L}_{ss}^{abc}\left[\mathbf{T}_{qd0}(\theta)\right]^{-1}\mathbf{i}_s^{qd0} + \left[\mathbf{T}_{qd0}(\theta)\right]\mathbf{L}_{sr}^{abc}\left[\mathbf{T}_{qd0}(\theta-\theta_r)\right]^{-1}\mathbf{i}_r^{qd0}$$

Induction Machine Modelling in Arbitrary $qd0$ RF

qd0 flux linkage relation

Stator

$$\boldsymbol{\lambda}_s^{qd0} = \underbrace{\left[\mathbf{T}_{qd0}(\theta)\right]\mathbf{L}_{ss}^{abc}\left[\mathbf{T}_{qd0}(\theta)\right]^{-1}}_{\mathbf{L}_{ss}^{qd0}} \mathbf{i}_s^{qd0} + \underbrace{\left[\mathbf{T}_{qd0}(\theta)\right]\mathbf{L}_{sr}^{abc}\left[\mathbf{T}_{qd0}(\theta-\theta_r)\right]^{-1}}_{\mathbf{L}_{sr}^{qd0}} \mathbf{i}_r^{qd0}$$

$$\Rightarrow \quad \boldsymbol{\lambda}_s^{qd0} = \mathbf{L}_{ss}^{qd0}\,\mathbf{i}_s^{qd0} + \mathbf{L}_{sr}^{qd0}\,\mathbf{i}_r^{qd0}$$

The inductance matrices L_{ss} and L_{sr} in *qd*0 reference frame need to be obtained.

Induction Machine Modelling in Arbitrary $qd0$ RF

qd0 flux linkage relation

Stator

where $\mathbf{L}_{ss}^{qd0} = \left[\mathbf{T}_{qd0}(\theta)\right]\mathbf{L}_{ss}^{abc}\left[\mathbf{T}_{qd0}(\theta)\right]^{-1}$ is the **stator self-inductance** matrix in $qd0$ reference frame

$$\mathbf{L}_{ss}^{qd0} = \frac{2}{3}\begin{bmatrix} \cos\theta & \cos\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta+\frac{2\pi}{3}\right) \\ \sin\theta & \sin\left(\theta-\frac{2\pi}{3}\right) & \sin\left(\theta+\frac{2\pi}{3}\right) \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} L_{ls}+L_{ss} & L_{sm} & L_{sm} \\ L_{sm} & L_{ls}+L_{ss} & L_{sm} \\ L_{sm} & L_{sm} & L_{ls}+L_{ss} \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta & 1 \\ \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(\theta-\frac{2\pi}{3}\right) & 1 \\ \cos\left(\theta+\frac{2\pi}{3}\right) & \sin\left(\theta+\frac{2\pi}{3}\right) & 1 \end{bmatrix}$$

$$\Rightarrow \quad \mathbf{L}_{ss}^{qd0} = \begin{bmatrix} L_{ls}+\frac{3}{2}L_{ss} & 0 & 0 \\ 0 & L_{ls}+\frac{3}{2}L_{ss} & 0 \\ 0 & 0 & L_{ls} \end{bmatrix}$$

Diagonal matrix and independent of θ.

Induction Machine Modelling in Arbitrary $qd0$ RF

qd0 **flux linkage relation**

Stator

and $\mathbf{L}_{sr}^{qd0} = \left[\mathbf{T}_{qd0}(\theta)\right]\mathbf{L}_{sr}^{abc}\left[\mathbf{T}_{qd0}(\theta-\theta_r)\right]^{-1}$ is the **stator-to-rotor mutual** inductance matrix.

where

$$\mathbf{L}_{sr}^{abc} = L_{sr}\begin{bmatrix} \cos\theta_r & \cos(\theta_r + 2\pi/3) & \cos(\theta_r - 2\pi/3) \\ \cos(\theta_r - 2\pi/3) & \cos\theta_r & \cos(\theta_r + 2\pi/3) \\ \cos(\theta_r + 2\pi/3) & \cos(\theta_r - 2\pi/3) & \cos\theta_r \end{bmatrix}$$

$$\Rightarrow \quad \mathbf{L}_{sr}^{qd0} = \begin{bmatrix} \frac{3}{2}L_{sr} & 0 & 0 \\ 0 & \frac{3}{2}L_{sr} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Diagonal matrix and independent of θ.

Induction Machine Modelling in Arbitrary $qd0$ RF

qd0 flux linkage relation

Rotor

- The **rotor flux linkage** equations in abc system is expressed as:

$$\boldsymbol{\lambda}_r^{abc} = \mathbf{L}_{rs}^{abc}\,\mathbf{i}_s^{abc} + \mathbf{L}_{rr}^{abc}\,\mathbf{i}_r^{abc}$$

- Applying the transformation $\mathbf{T}_{qd0}(\theta)$ and $\mathbf{T}_{qd0}(\theta-\theta_r)$ to the stator and rotor quantities yields

$$\left[\mathbf{T}_{qd0}(\theta-\theta_r)\right]^{-1}\boldsymbol{\lambda}_r^{qd0} = \mathbf{L}_{rs}^{abc}\left[\mathbf{T}_{qd0}(\theta)\right]^{-1}\mathbf{i}_s^{qd0} + \mathbf{L}_{rr}^{abc}\left[\mathbf{T}_{qd0}(\theta-\theta_r)\right]^{-1}\mathbf{i}_r^{qd0}$$

- Multiplying both sides by $\mathbf{T}_{qd0}(\theta-\theta_r)$ yields

$$\boldsymbol{\lambda}_r^{qd0} = \left[\mathbf{T}_{qd0}(\theta-\theta_r)\right]\mathbf{L}_{rs}^{abc}\left[\mathbf{T}_{qd0}(\theta)\right]^{-1}\mathbf{i}_s^{qd0} + \left[\mathbf{T}_{qd0}(\theta-\theta_r)\right]\mathbf{L}_{rr}^{abc}\left[\mathbf{T}_{qd0}(\theta-\theta_r)\right]^{-1}\mathbf{i}_r^{qd0}$$

Induction Machine Modelling in Arbitrary $qd0$ RF

qd0 flux linkage relation

Rotor

$$\boldsymbol{\lambda}_r^{qd0} = \underbrace{\left[\mathbf{T}_{qd0}(\theta-\theta_r)\right]\mathbf{L}_{rs}^{abc}\left[\mathbf{T}_{qd0}(\theta)\right]^{-1}}_{\mathbf{L}_{rs}^{qd0}}\mathbf{i}_s^{qd0} + \underbrace{\left[\mathbf{T}_{qd0}(\theta-\theta_r)\right]\mathbf{L}_{rr}^{abc}\left[\mathbf{T}_{qd0}(\theta-\theta_r)\right]^{-1}}_{\mathbf{L}_{rr}^{qd0}}\mathbf{i}_r^{qd0}$$

$$\Rightarrow \quad \boldsymbol{\lambda}_r^{qd0} = \mathbf{L}_{rs}^{qd0}\,\mathbf{i}_s^{qd0} + \mathbf{L}_{rr}^{qd0}\,\mathbf{i}_r^{qd0}$$

The inductance matrices L_{rs} and L_{rr} in *qd*0 reference frame need to be obtained.

Induction Machine Modelling in Arbitrary $qd0$ RF

qd0 flux linkage relation

Rotor

where $\mathbf{L}_{rr}^{qd0} = \left[\mathbf{T}_{qd0}(\theta-\theta_r)\right]\mathbf{L}_{rr}^{abc}\left[\mathbf{T}_{qd0}(\theta-\theta_r)\right]^{-1}$ is the **rotor self-inductance** matrix in $qd0$ reference frame

$$\mathbf{L}_{rr}^{qd0} = \begin{bmatrix} L_{lr} + \frac{3}{2}L_{rr} & 0 & 0 \\ 0 & L_{lr} + \frac{3}{2}L_{rr} & 0 \\ 0 & 0 & L_{lr} \end{bmatrix}$$

Diagonal matrix and independent of θ.

Induction Machine Modelling in Arbitrary $qd0$ RF

qd0 flux linkage relation

Rotor

and $\mathbf{L}_{rs}^{qd0} = \left[\mathbf{T}_{qd0}(\theta-\theta_r)\right]\mathbf{L}_{rs}^{abc}\left[\mathbf{T}_{qd0}(\theta)\right]^{-1}$ is the **rotor-to-stator mutual** inductance matrix.

$$\mathbf{L}_{rs}^{qd0} = \begin{bmatrix} \frac{3}{2}L_{sr} & 0 & 0 \\ 0 & \frac{3}{2}L_{sr} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Diagonal matrix and independent of θ.

Induction Machine Modelling in Arbitrary $qd0$ RF

Stator & Rotor

qd0 flux linkage relation

Therefore the flux linkage relation of induction machines in $qd0$ reference frame is expressed as

$$\begin{bmatrix} \lambda_{qs} \\ \lambda_{ds} \\ \lambda_{0s} \\ \hline \lambda_{qr} \\ \lambda_{dr} \\ \lambda_{0r} \end{bmatrix} = \left[\begin{array}{ccc|ccc} L_{ls}+\frac{3}{2}L_{ss} & 0 & 0 & \frac{3}{2}L_{sr} & 0 & 0 \\ 0 & L_{ls}+\frac{3}{2}L_{ss} & 0 & 0 & \frac{3}{2}L_{sr} & 0 \\ 0 & 0 & L_{ls} & 0 & 0 & 0 \\ \hline \frac{3}{2}L_{sr} & 0 & 0 & L_{lr}+\frac{3}{2}L_{rr} & 0 & 0 \\ 0 & \frac{3}{2}L_{sr} & 0 & 0 & L_{lr}+\frac{3}{2}L_{rr} & 0 \\ 0 & 0 & 0 & 0 & 0 & L_{lr} \end{array}\right] \begin{bmatrix} i_{qs} \\ i_{ds} \\ i_{0s} \\ \hline i_{qr} \\ i_{dr} \\ i_{0r} \end{bmatrix}$$

It is required to refer all rotor quantities to the stator side.

Induction Machine Modelling in Arbitrary $qd0$ RF

qd0 flux linkage relation referred to stator side

Stator & Rotor

$$\begin{bmatrix} \lambda_{qs} \\ \lambda_{ds} \\ \lambda_{0s} \\ \hline \lambda'_{qr} \\ \lambda'_{dr} \\ \lambda'_{0r} \end{bmatrix} = \left[\begin{array}{ccc|ccc} L_{ls}+\frac{3}{2}L_{ss} & 0 & 0 & \frac{3}{2}\left(\frac{N_s}{N_r}\right)L_{sr} & 0 & 0 \\ 0 & L_{ls}+\frac{3}{2}L_{ss} & 0 & 0 & \frac{3}{2}\left(\frac{N_s}{N_r}\right)L_{sr} & 0 \\ 0 & 0 & L_{ls} & 0 & 0 & 0 \\ \hline \frac{3}{2}\left(\frac{N_s}{N_r}\right)L_{sr} & 0 & 0 & \left(\frac{N_s}{N_r}\right)^2\left(L_{lr}+\frac{3}{2}L_{rr}\right) & 0 & 0 \\ 0 & \frac{3}{2}\left(\frac{N_s}{N_r}\right)L_{sr} & 0 & 0 & \left(\frac{N_s}{N_r}\right)^2\left(L_{lr}+\frac{3}{2}L_{rr}\right) & 0 \\ 0 & 0 & 0 & 0 & 0 & \left(\frac{N_s}{N_r}\right)^2 L_{lr} \end{array}\right] \begin{bmatrix} i_{qs} \\ i_{ds} \\ i_{0s} \\ \hline i'_{qr} \\ i'_{dr} \\ i'_{0r} \end{bmatrix}$$

The **magnetizing inductance**, L_m, on the stator side is expressed as

$$L_m = \frac{3}{2}L_{ss} = \frac{3}{2}\frac{N_s}{N_r}L_{sr} = \frac{3}{2}\left(\frac{N_s}{N_r}\right)^2 L_{rr}$$

Why?

Induction Machine Modelling in Arbitrary $qd0$ RF

qd0 flux linkage relation referred to stator side

Stator & Rotor

Therefore *qd*0 flux linkage relation is expressed as

$$\begin{bmatrix} \lambda_{qs} \\ \lambda_{ds} \\ \lambda_{0s} \\ \hline \lambda'_{qr} \\ \lambda'_{dr} \\ \lambda'_{0r} \end{bmatrix} = \left[\begin{array}{ccc|ccc} L_{ls}+L_m & 0 & 0 & L_m & 0 & 0 \\ 0 & L_{ls}+L_m & 0 & 0 & L_m & 0 \\ 0 & 0 & L_{ls} & 0 & 0 & 0 \\ \hline L_m & 0 & 0 & L'_{lr}+L_m & 0 & 0 \\ 0 & L_m & 0 & 0 & L'_{lr}+L_m & 0 \\ 0 & 0 & 0 & 0 & 0 & L'_{lr} \end{array}\right] \begin{bmatrix} i_{qs} \\ i_{ds} \\ i_{0s} \\ \hline i'_{qr} \\ i'_{dr} \\ i'_{0r} \end{bmatrix}$$

where

$$L_m = \tfrac{3}{2} L_{ss} = \tfrac{3}{2}\frac{N_s}{N_r} L_{sr} = \tfrac{3}{2}\left(\frac{N_s}{N_r}\right)^2 L_{rr}$$

$$L'_{lr} = \left(\frac{N_s}{N_r}\right)^2 L_{lr}$$

Induction Machine Modelling in Arbitrary $qd0$ RF

- Lets prove the following relation

$$L_m = \tfrac{3}{2} L_{ss} = \tfrac{3}{2} \frac{N_s}{N_r} L_{sr} = \tfrac{3}{2} \left(\frac{N_s}{N_r}\right)^2 L_{rr}$$

- Remind the aforementioned relations

$$L_{ss} = N_s^2 P_g \qquad L_{rr} = N_r^2 P_g \qquad L_{sr} = N_s N_r P_g$$

- Therefore using the above relations:

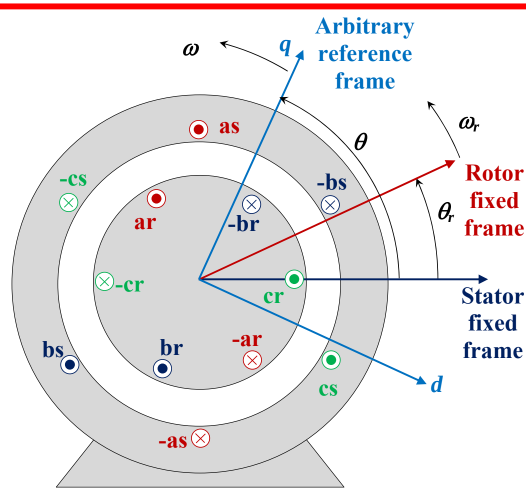
$$L_{ss} = N_s^2 P_g = \left(\frac{N_s}{N_r}\right)^2 N_r^2 P_g = \left(\frac{N_s}{N_r}\right)^2 L_{rr}$$

$$L_{ss} = N_s^2 P_g = \left(\frac{N_s}{N_r}\right) N_s N_r P_g = \left(\frac{N_s}{N_r}\right) L_{sr}$$

Induction Machine Modelling in Arbitrary $qd0$ RF

Therefore $qd0$ model is expressed as

$$
\begin{bmatrix} v_{qs} \\ v_{ds} \\ v_{0s} \\ \hline v'_{qr} \\ v'_{dr} \\ v'_{0r} \end{bmatrix}
=
\left[\begin{array}{ccc|ccc}
r_s + p(L_{ls}+L_m) & \omega(L_{ls}+L_m) & 0 & pL_m & \omega L_m & 0 \\
-\omega(L_{ls}+L_m) & r_s + p(L_{ls}+L_m) & 0 & -\omega L_m & pL_m & 0 \\
0 & 0 & r_s + pL_{ls} & 0 & 0 & 0 \\
\hline
pL_m & (\omega-\omega_r)L_m & 0 & r'_r + p(L'_{lr}+L_m) & (\omega-\omega_r)(L'_{lr}+L_m) & 0 \\
-(\omega-\omega_r)L_m & pL_m & 0 & -(\omega-\omega_r)(L'_{lr}+L_m) & r'_r + p(L'_{lr}+L_m) & 0 \\
0 & 0 & 0 & 0 & 0 & r'_r + pL'_{lr}
\end{array}\right]
\begin{bmatrix} i_{qs} \\ i_{ds} \\ i_{0s} \\ \hline i'_{qr} \\ i'_{dr} \\ i'_{0r} \end{bmatrix}
$$



Induction Machine Modelling in Arbitrary $qd0$ RF

- Using the **reactances** instead of the **inductances** in the **stator** voltage and flux linkage equations yields:

$$\begin{cases} v_{qs} = r_s\, i_{qs} + \frac{1}{\omega_b}\dfrac{d\psi_{qs}}{dt} + \frac{\omega}{\omega_b}\psi_{ds} \\ v_{ds} = r_s\, i_{ds} + \frac{1}{\omega_b}\dfrac{d\psi_{ds}}{dt} - \frac{\omega}{\omega_b}\psi_{qs} \end{cases}$$

$$\psi_{qs} = \omega_b\, \lambda_{qs}$$

$$\psi_{ds} = \omega_b\, \lambda_{ds}$$

$$\begin{cases} \psi_{qs} = x_{ls}\, i_{qs} + x_m (i_{qs} + i'_{qr}) \\ \psi_{ds} = x_{ls}\, i_{ds} + x_m (i_{ds} + i'_{dr}) \end{cases}$$

$$x_{ls} = \omega_b\, L_{ls}$$

$$x_m = \omega_b\, L_m$$

Induction Machine Modelling in Arbitrary $qd0$ RF

- Similarly using the reactances instead of the inductances in the rotor voltage and flux linkage equations (referred to stator) yields:

$$v'_{qr} = r'_r i'_{qr} + \frac{1}{\omega_b}\frac{d\psi'_{qr}}{dt} + \frac{\omega-\omega_r}{\omega_b}\psi'_{dr}$$

$$v'_{dr} = r'_r i'_{dr} + \frac{1}{\omega_b}\frac{d\psi'_{dr}}{dt} - \frac{\omega-\omega_r}{\omega_b}\psi'_{qr}$$

$$\psi'_{qr} = \omega_b \lambda'_{qr}$$

$$\psi'_{dr} = \omega_b \lambda'_{dr}$$

$$\psi'_{qr} = x'_{lr} i'_{qr} + x_m(i_{qs} + i'_{qr})$$

$$\psi'_{dr} = x'_{lr} i'_{dr} + x_m(i_{ds} + i'_{dr})$$

$$x'_{lr} = \omega_b L'_{lr}$$

$$x_m = \omega_b L_m$$

Induction Machine Modelling in Arbitrary *qd*0 RF

q-axis equivalent circuit

Stator

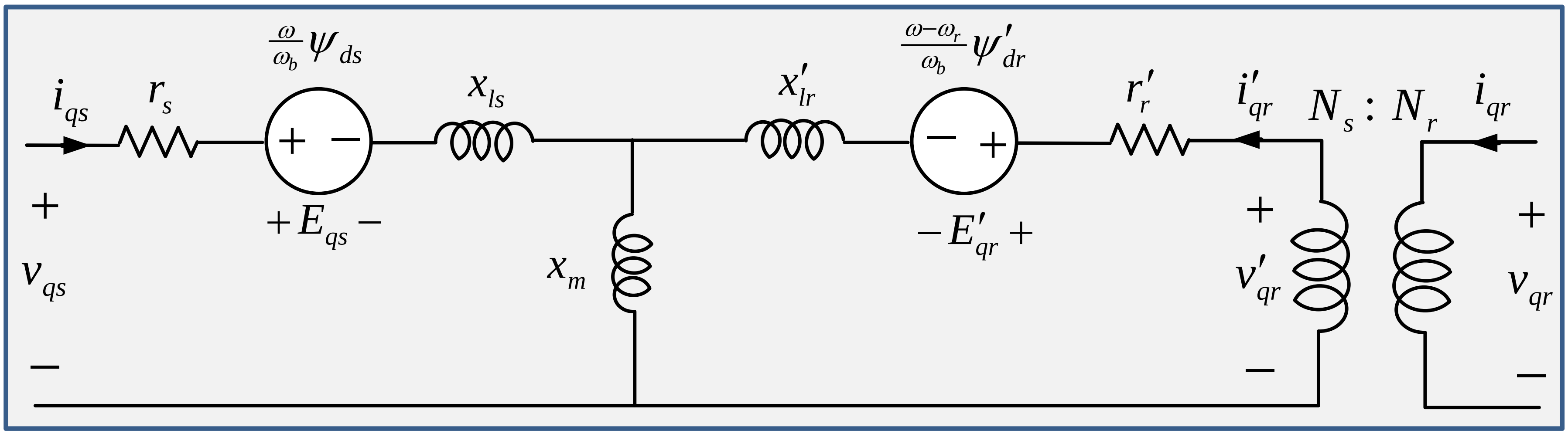
$$v_{qs} = r_s\, i_{qs} + \tfrac{1}{\omega_b}\frac{d\psi_{qs}}{dt} + \tfrac{\omega}{\omega_b}\psi_{ds}$$

$$\psi_{qs} = x_{ls}\, i_{qs} + x_m(i_{qs} + i'_{qr})$$

Rotor

$$v'_{qr} = r'_r\, i'_{qr} + \tfrac{1}{\omega_b}\frac{d\psi'_{qr}}{dt} + \tfrac{\omega-\omega_r}{\omega_b}\psi'_{dr}$$

$$\psi'_{qr} = x'_{lr}\, i'_{qr} + x_m(i_{qs} + i'_{qr})$$



Induction Machine Modelling in Arbitrary *qd*0 RF

d-axis equivalent circuit

Stator

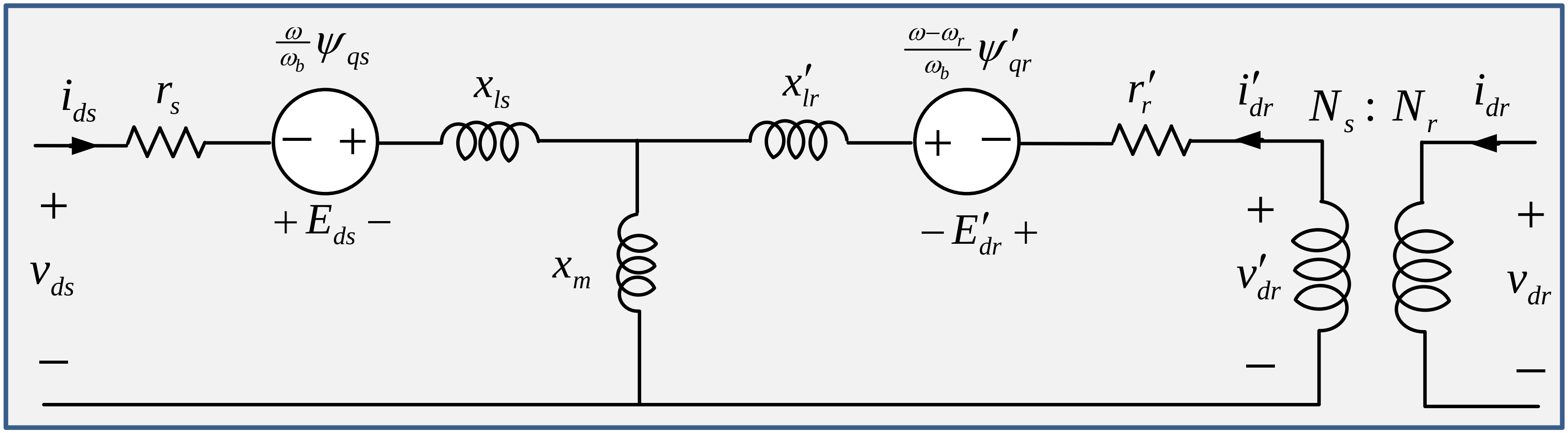
$$v_{ds} = r_s\, i_{ds} + \frac{1}{\omega_b}\frac{d\psi_{ds}}{dt} - \frac{\omega}{\omega_b}\psi_{qs}$$

$$\psi_{ds} = x_{ls}\, i_{ds} + x_m(i_{ds} + i'_{dr})$$

Rotor

$$v'_{dr} = r'_r\, i'_{dr} + \frac{1}{\omega_b}\frac{d\psi'_{dr}}{dt} - \frac{\omega-\omega_r}{\omega_b}\psi'_{qr}$$

$$\psi'_{dr} = x'_{lr}\, i'_{dr} + x_m(i_{ds} + i'_{dr})$$



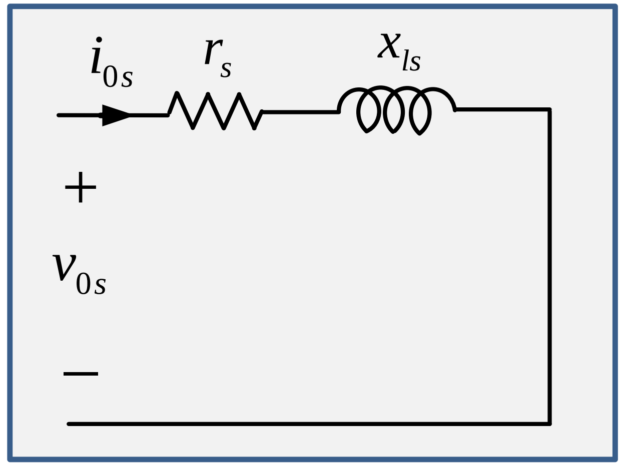
Induction Machine Modelling in Arbitrary $qd0$ RF

0 component equivalent circuit

Stator

$$v_{0s} = r_s\, i_{0s} + \frac{1}{\omega_b}\frac{d\psi_{0s}}{dt}$$

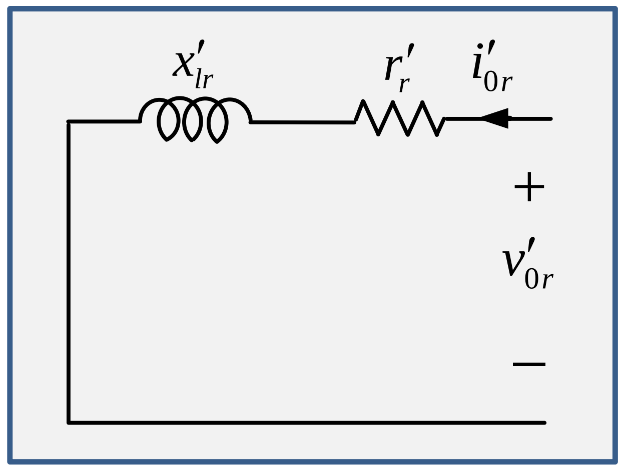
$$\psi_{0s} = x_{ls}\, i_{0s}$$



Rotor

$$v'_{0r} = r'_r\, i'_{0r} + \frac{1}{\omega_b}\frac{d\psi'_{0r}}{dt}$$

$$\psi'_{0r} = x'_{lr}\, i'_{0r}$$



Induction Machine Modelling in Arbitrary *qd*0 RF

Torque Equation

- The instantaneous input power in *abc* system is obtained as

$$P_{in} = v_{as}\, i_{as} + v_{bs}\, i_{bs} + v_{cs}\, i_{cs} + v'_{ar}\, i'_{ar} + v'_{br}\, i'_{br} + v'_{cr}\, i'_{cr}$$

- In terms of qd0 quantities, the instantaneous input power is

$$P_{in} = \tfrac{3}{2}\left(v_{qs}\, i_{qs} + v_{ds}\, i_{ds} + 2v_{0s}\, i_{0s} + v'_{qr}\, i'_{qr} + v'_{dr}\, i'_{dr} + 2v'_{0r}\, i'_{0r}\right)$$

- Substituting the voltage equations in the above expression yields

$$\begin{aligned} P_{in} = \tfrac{3}{2}\Big(& r_s\, i_{qs}^2 + i_{qs}\, p\lambda_{qs} + \omega\lambda_{ds} i_{qs} + r_s\, i_{ds}^2 + i_{ds}\, p\lambda_{ds} - \omega\lambda_{qs} i_{ds} + 2r_s\, i_{0s}^2 + 2\, i_{0s}\, p\lambda_{0s} \\ & + r'_r\, i'^2_{qr} + i'_{qr}\, p\lambda'_{qr} + (\omega - \omega_r)\lambda'_{dr} i'_{qr} + r'_r\, i'^2_{dr} + i'_{dr}\, p\lambda'_{dr} - (\omega - \omega_r)\lambda'_{qr} i'_{dr} \\ & + 2r'_r\, i'^2_{0r} + 2\, i'_{0r}\, p\lambda'_{0r} \Big) \end{aligned}$$

Induction Machine Modelling in Arbitrary $qd0$ RF

Torque Equation

$$P_{in} = \tfrac{3}{2}\Big(r_s\, i_{qs}^2 + i_{qs}\, p\lambda_{qs} + \omega\lambda_{ds} i_{qs} + r_s\, i_{ds}^2 + i_{ds}\, p\lambda_{ds} - \omega\lambda_{qs} i_{ds} + 2r_s\, i_{0s}^2 + 2\, i_{0s}\, p\lambda_{0s}$$
$$+ r_r'\, i_{qr}'^2 + i_{qr}'\, p\lambda_{qr}' + (\omega - \omega_r)\lambda_{dr}' i_{qr}' + r_r'\, i_{dr}'^2 + i_{dr}'\, p\lambda_{dr}' - (\omega - \omega_r)\lambda_{qr}' i_{dr}'$$
$$+ 2r_r'\, i_{0r}'^2 + 2\, i_{0r}'\, p\lambda_{0r}'\Big)$$

- The **ohmic losses** (ri^2 terms) and the **rate of exchange of magnetic field energy** between windings ($ip\lambda$ terms) **do not contribute** to electromagnetic (EM) torque development.

- Hence the **power terms contributed** to EM torque generation are:

$$P_T = \tfrac{3}{2}\Big[\omega\big(\lambda_{ds} i_{qs} - \lambda_{qs} i_{ds}\big) + (\omega - \omega_r)\big(\lambda_{dr}' i_{qr}' - \lambda_{qr}' i_{dr}'\big)\Big]$$

Induction Machine Modelling in Arbitrary $qd0$ RF

Torque Equation

$$P_T = \frac{3}{2}\left[\omega\left(\lambda_{ds} i_{qs} - \lambda_{qs} i_{ds}\right) + (\omega - \omega_r)\left(\lambda'_{dr} i'_{qr} - \lambda'_{qr} i'_{dr}\right)\right]$$

- The electromagnetic torque is obtained by dividing the above power by the **mechanical speed**:

$$T_{em} = \frac{3}{2}\frac{P}{2\omega_r}\left[\omega\left(\lambda_{ds} i_{qs} - \lambda_{qs} i_{ds}\right) + (\omega - \omega_r)\left(\lambda'_{dr} i'_{qr} - \lambda'_{qr} i'_{dr}\right)\right]$$

where P is the number of poles.

- It can be shown that:

$$\lambda_{ds} i_{qs} - \lambda_{qs} i_{ds} = -(\lambda'_{dr} i'_{qr} - \lambda'_{qr} i'_{dr}) = L_m (i'_{dr} i_{qs} - i'_{qr} i_{ds})$$

Induction Machine Modelling in Arbitrary $qd0$ RF

Torque Equation

$$T_{em} = \frac{3}{2}\frac{P}{2\omega_r}\left[\omega\left(\lambda_{ds}i_{qs} - \lambda_{qs}i_{ds}\right) + (\omega - \omega_r)\left(\lambda'_{dr}i'_{qr} - \lambda'_{qr}i'_{dr}\right)\right]$$

$$\lambda_{ds}i_{qs} - \lambda_{qs}i_{ds} = -(\lambda'_{dr}i'_{qr} - \lambda'_{qr}i'_{dr}) = L_m(i'_{dr}i_{qs} - i'_{qr}i_{ds})$$

- Using the above relations, the following expressions for the electromagnetic torque calculation are obtained:

$$T_{em} = \frac{3}{2}\frac{P}{2}\left(\lambda'_{qr}i'_{dr} - \lambda'_{dr}i'_{qr}\right)$$

$$T_{em} = \frac{3}{2}\frac{P}{2}\left(\lambda_{ds}i_{qs} - \lambda_{qs}i_{ds}\right)$$

$$T_{em} = \frac{3}{2}\frac{P}{2}L_m\left(i'_{dr}i_{qs} - i'_{qr}i_{ds}\right)$$

Induction Machine Modelling in Arbitrary $qd0$ RF

Mechanical Dynamic Equation

- Based on Newton's 2nd law for rotational movement, $\sum T = J\frac{d\omega_{rm}}{dt}$ the **mechanical dynamic equation** is obtained:

Motoring Mode

$$T_{em} - T_{mech} - T_{damp} = J\frac{d\omega_{rm}}{dt}$$

Generating Mode

$$T_{em} + T_{mech} - T_{damp} = J\frac{d\omega_{rm}}{dt}$$

where T_{mech} is the externally-applied mechanical torque, T_{damp} is the damping torque, J is the moment of inertia and ω_{rm} is the mechanical rotational velocity.

Induction Machine Modelling in $qd0$ RF

- From the dynamic equations in *qd*0 reference frame with **arbitrary rotating speed**, the equations in the same reference frame **with other rotating speed** can be obtained:

Stationary Reference Frame → $\omega = 0$

Fixed-to-rotor Reference Frame → $\omega = \omega_r$

Synchronous Reference Frame → $\omega = \omega_e$

IM Model in $qd0$ RF

Arbitrary rotating speed ω

$$v_{qs} = r_s\, i_{qs} + \tfrac{1}{\omega_b}\frac{d\psi_{qs}}{dt} + \tfrac{\omega}{\omega_b}\psi_{ds}$$

$$v_{ds} = r_s\, i_{ds} + \tfrac{1}{\omega_b}\frac{d\psi_{ds}}{dt} - \tfrac{\omega}{\omega_b}\psi_{qs}$$

$$v_{0s} = r_s\, i_{0s} + \tfrac{1}{\omega_b}\frac{d\psi_{0s}}{dt}$$

$$v'_{qr} = r'_r\, i'_{qr} + \tfrac{1}{\omega_b}\frac{d\psi'_{qr}}{dt} + \tfrac{\omega-\omega_r}{\omega_b}\psi'_{dr}$$

$$v'_{dr} = r'_r\, i'_{dr} + \tfrac{1}{\omega_b}\frac{d\psi'_{dr}}{dt} - \tfrac{\omega-\omega_r}{\omega_b}\psi'_{qr}$$

$$v'_{0r} = r'_r\, i'_{0r} + \tfrac{1}{\omega_b}\frac{d\psi'_{0r}}{dt}$$

$$\begin{bmatrix}\psi_{qs}\\ \psi_{ds}\\ \psi_{0s}\\ \psi'_{qr}\\ \psi'_{dr}\\ \psi'_{0r}\end{bmatrix} = \begin{bmatrix} x_{ls}+x_m & 0 & 0 & x_m & 0 & 0\\ 0 & x_{ls}+x_m & 0 & 0 & x_m & 0\\ 0 & 0 & x_{ls} & 0 & 0 & 0\\ x_m & 0 & 0 & x'_{lr}+x_m & 0 & 0\\ 0 & x_m & 0 & 0 & x'_{lr}+x_m & 0\\ 0 & 0 & 0 & 0 & 0 & x'_{lr}\end{bmatrix}\begin{bmatrix} i_{qs}\\ i_{ds}\\ i_{0s}\\ i'_{qr}\\ i'_{dr}\\ i'_{0r}\end{bmatrix}$$

$$T_{em} - T_{mech} - T_{damp} = J\frac{d\omega_{rm}}{dt}$$

$$T_{em} = \tfrac{3}{2}\tfrac{P}{2\omega_b}\left(\psi'_{qr} i'_{dr} - \psi'_{dr} i'_{qr}\right) = \tfrac{3}{2}\tfrac{P}{2\omega_b}\left(\psi_{ds} i_{qs} - \psi_{qs} i_{ds}\right) = \tfrac{3}{2}\tfrac{P}{2\omega_b} x_m\left(i'_{dr} i_{qs} - i'_{qr} i_{ds}\right)$$

IM Model in *qd*0 RF

Stationary Reference Frame $\omega = 0$

$$v_{qs} = r_s\, i_{qs} + \frac{1}{\omega_b}\frac{d\psi_{qs}}{dt}$$

$$v_{ds} = r_s\, i_{ds} + \frac{1}{\omega_b}\frac{d\psi_{ds}}{dt}$$

$$v_{0s} = r_s\, i_{0s} + \frac{1}{\omega_b}\frac{d\psi_{0s}}{dt}$$

$$v'_{qr} = r'_r\, i'_{qr} + \frac{1}{\omega_b}\frac{d\psi'_{qr}}{dt} - \frac{\omega_r}{\omega_b}\psi'_{dr}$$

$$v'_{dr} = r'_r\, i'_{dr} + \frac{1}{\omega_b}\frac{d\psi'_{dr}}{dt} + \frac{\omega_r}{\omega_b}\psi'_{qr}$$

$$v'_{0r} = r'_r\, i'_{0r} + \frac{1}{\omega_b}\frac{d\psi'_{0r}}{dt}$$

$$\begin{bmatrix} \psi_{qs} \\ \psi_{ds} \\ \psi_{0s} \\ \psi'_{qr} \\ \psi'_{dr} \\ \psi'_{0r} \end{bmatrix} = \begin{bmatrix} x_{ls}+x_m & 0 & 0 & x_m & 0 & 0 \\ 0 & x_{ls}+x_m & 0 & 0 & x_m & 0 \\ 0 & 0 & x_{ls} & 0 & 0 & 0 \\ x_m & 0 & 0 & x'_{lr}+x_m & 0 & 0 \\ 0 & x_m & 0 & 0 & x'_{lr}+x_m & 0 \\ 0 & 0 & 0 & 0 & 0 & x'_{lr} \end{bmatrix} \begin{bmatrix} i_{qs} \\ i_{ds} \\ i_{0s} \\ i'_{qr} \\ i'_{dr} \\ i'_{0r} \end{bmatrix}$$

$$T_{em} - T_{mech} - T_{damp} = J\frac{d\omega_{rm}}{dt}$$

$$T_{em} = \frac{3}{2}\frac{P}{2\omega_b}\left(\psi'_{qr} i'_{dr} - \psi'_{dr} i'_{qr}\right) = \frac{3}{2}\frac{P}{2\omega_b}\left(\psi_{ds} i_{qs} - \psi_{qs} i_{ds}\right) = \frac{3}{2}\frac{P}{2\omega_b} x_m\left(i'_{dr} i_{qs} - i'_{qr} i_{ds}\right)$$

IM Model in *qd*0 RF

Fixed-to-rotor Reference Frame $\omega = \omega_r$

$$v_{qs} = r_s\, i_{qs} + \tfrac{1}{\omega_b}\frac{d\psi_{qs}}{dt} + \tfrac{\omega_r}{\omega_b}\psi_{ds}$$

$$v_{ds} = r_s\, i_{ds} + \tfrac{1}{\omega_b}\frac{d\psi_{ds}}{dt} - \tfrac{\omega_r}{\omega_b}\psi_{qs}$$

$$v_{0s} = r_s\, i_{0s} + \tfrac{1}{\omega_b}\frac{d\psi_{0s}}{dt}$$

$$v'_{qr} = r'_r\, i'_{qr} + \tfrac{1}{\omega_b}\frac{d\psi'_{qr}}{dt}$$

$$v'_{dr} = r'_r\, i'_{dr} + \tfrac{1}{\omega_b}\frac{d\psi'_{dr}}{dt}$$

$$v'_{0r} = r'_r\, i'_{0r} + \tfrac{1}{\omega_b}\frac{d\psi'_{0r}}{dt}$$

$$\begin{bmatrix}\psi_{qs}\\ \psi_{ds}\\ \psi_{0s}\\ \psi'_{qr}\\ \psi'_{dr}\\ \psi'_{0r}\end{bmatrix} = \begin{bmatrix} x_{ls}+x_m & 0 & 0 & x_m & 0 & 0\\ 0 & x_{ls}+x_m & 0 & 0 & x_m & 0\\ 0 & 0 & x_{ls} & 0 & 0 & 0\\ x_m & 0 & 0 & x'_{lr}+x_m & 0 & 0\\ 0 & x_m & 0 & 0 & x'_{lr}+x_m & 0\\ 0 & 0 & 0 & 0 & 0 & x'_{lr}\end{bmatrix}\begin{bmatrix} i_{qs}\\ i_{ds}\\ i_{0s}\\ i'_{qr}\\ i'_{dr}\\ i'_{0r}\end{bmatrix}$$

$$T_{em} - T_{mech} - T_{damp} = J\frac{d\omega_{rm}}{dt}$$

$$T_{em} = \tfrac{3}{2}\tfrac{P}{2\omega_b}\left(\psi'_{qr} i'_{dr} - \psi'_{dr} i'_{qr}\right) = \tfrac{3}{2}\tfrac{P}{2\omega_b}\left(\psi_{ds} i_{qs} - \psi_{qs} i_{ds}\right) = \tfrac{3}{2}\tfrac{P}{2\omega_b}x_m\left(i'_{dr} i_{qs} - i'_{qr} i_{ds}\right)$$

IM Model in *qd*0 RF

Synchronous Reference Frame $\omega = \omega_e$

$$v_{qs} = r_s\, i_{qs} + \tfrac{1}{\omega_b}\frac{d\psi_{qs}}{dt} + \tfrac{\omega_e}{\omega_b}\psi_{ds}$$

$$v_{ds} = r_s\, i_{ds} + \tfrac{1}{\omega_b}\frac{d\psi_{ds}}{dt} - \tfrac{\omega_e}{\omega_b}\psi_{qs}$$

$$v_{0s} = r_s\, i_{0s} + \tfrac{1}{\omega_b}\frac{d\psi_{0s}}{dt}$$

$$v'_{qr} = r'_r\, i'_{qr} + \tfrac{1}{\omega_b}\frac{d\psi'_{qr}}{dt} + \tfrac{\omega_e-\omega_r}{\omega_b}\psi'_{dr}$$

$$v'_{dr} = r'_r\, i'_{dr} + \tfrac{1}{\omega_b}\frac{d\psi'_{dr}}{dt} - \tfrac{\omega_e-\omega_r}{\omega_b}\psi'_{qr}$$

$$v'_{0r} = r'_r\, i'_{0r} + \tfrac{1}{\omega_b}\frac{d\psi'_{0r}}{dt}$$

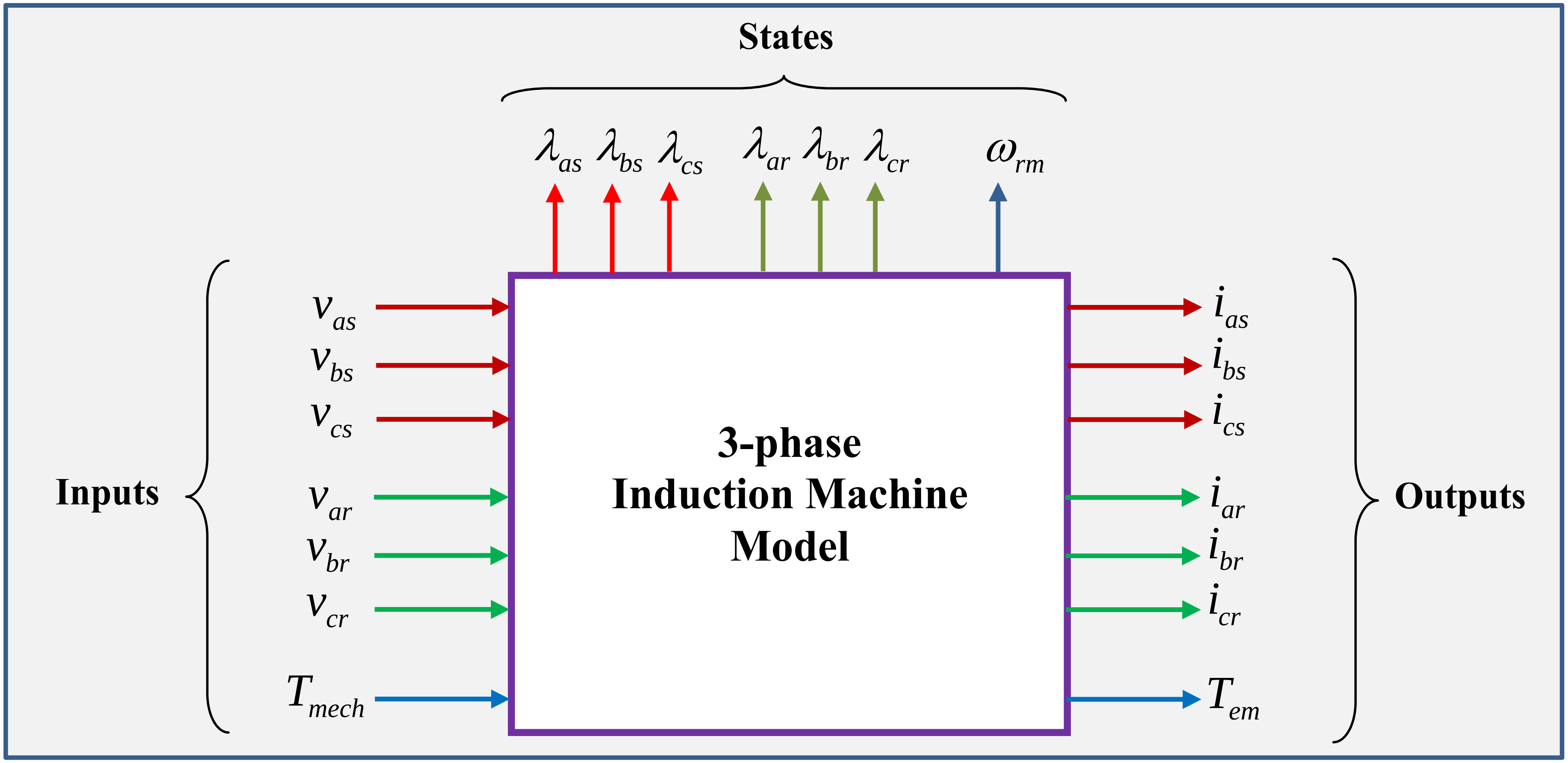
$$\begin{bmatrix}\psi_{qs}\\ \psi_{ds}\\ \psi_{0s}\\ \psi'_{qr}\\ \psi'_{dr}\\ \psi'_{0r}\end{bmatrix} = \begin{bmatrix} x_{ls}+x_m & 0 & 0 & x_m & 0 & 0\\ 0 & x_{ls}+x_m & 0 & 0 & x_m & 0\\ 0 & 0 & x_{ls} & 0 & 0 & 0\\ x_m & 0 & 0 & x'_{lr}+x_m & 0 & 0\\ 0 & x_m & 0 & 0 & x'_{lr}+x_m & 0\\ 0 & 0 & 0 & 0 & 0 & x'_{lr}\end{bmatrix}\begin{bmatrix} i_{qs}\\ i_{ds}\\ i_{0s}\\ i'_{qr}\\ i'_{dr}\\ i'_{0r}\end{bmatrix}$$

$$T_{em} - T_{mech} - T_{damp} = J\frac{d\omega_{rm}}{dt}$$

$$T_{em} = \tfrac{3}{2}\tfrac{P}{2\omega_b}\left(\psi'_{qr} i'_{dr} - \psi'_{dr} i'_{qr}\right) = \tfrac{3}{2}\tfrac{P}{2\omega_b}\left(\psi_{ds} i_{qs} - \psi_{qs} i_{ds}\right) = \tfrac{3}{2}\tfrac{P}{2\omega_b} x_m\left(i'_{dr} i_{qs} - i'_{qr} i_{ds}\right)$$

Simulation of Induction Machines

- To simulate the induction machines the inputs, outputs and states should be defined



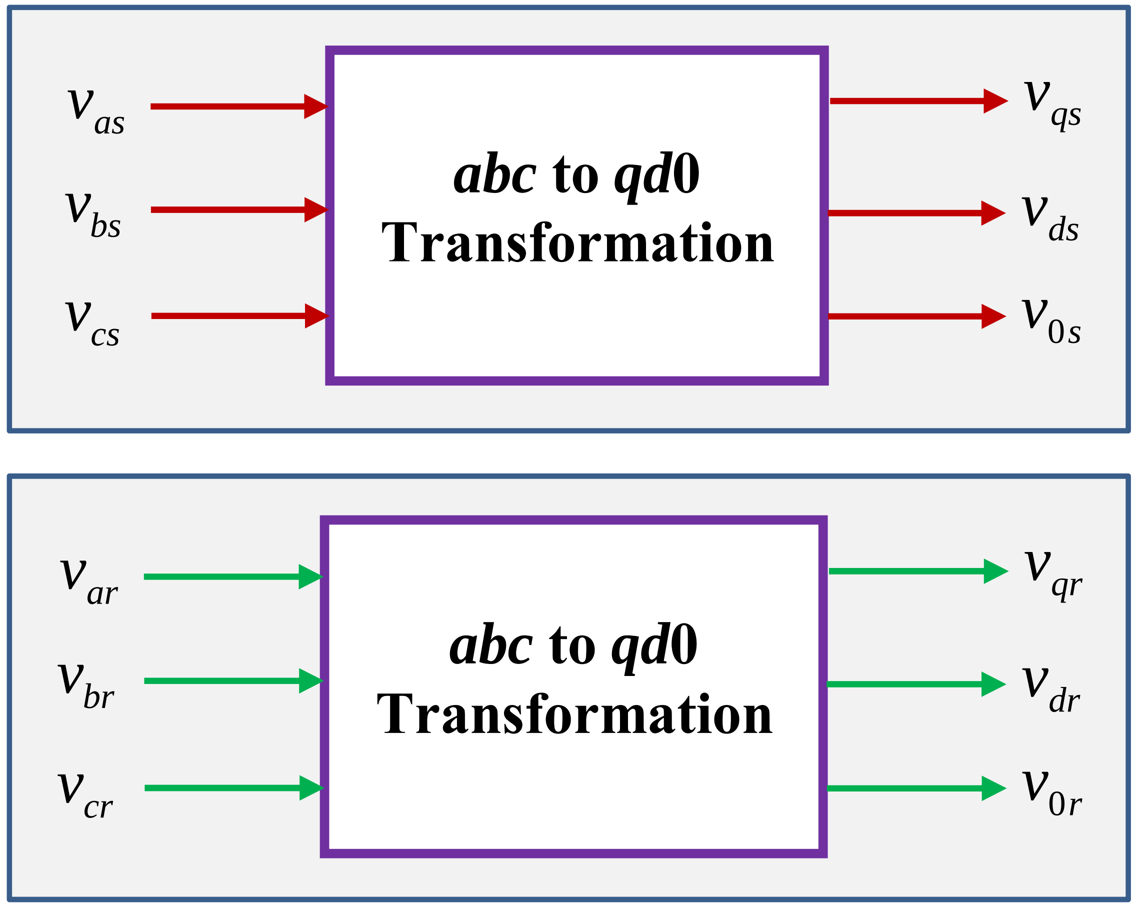
Simulation of Induction Machines

Sub-systems: Transformation from *abc* to *qd*0

Stator voltages

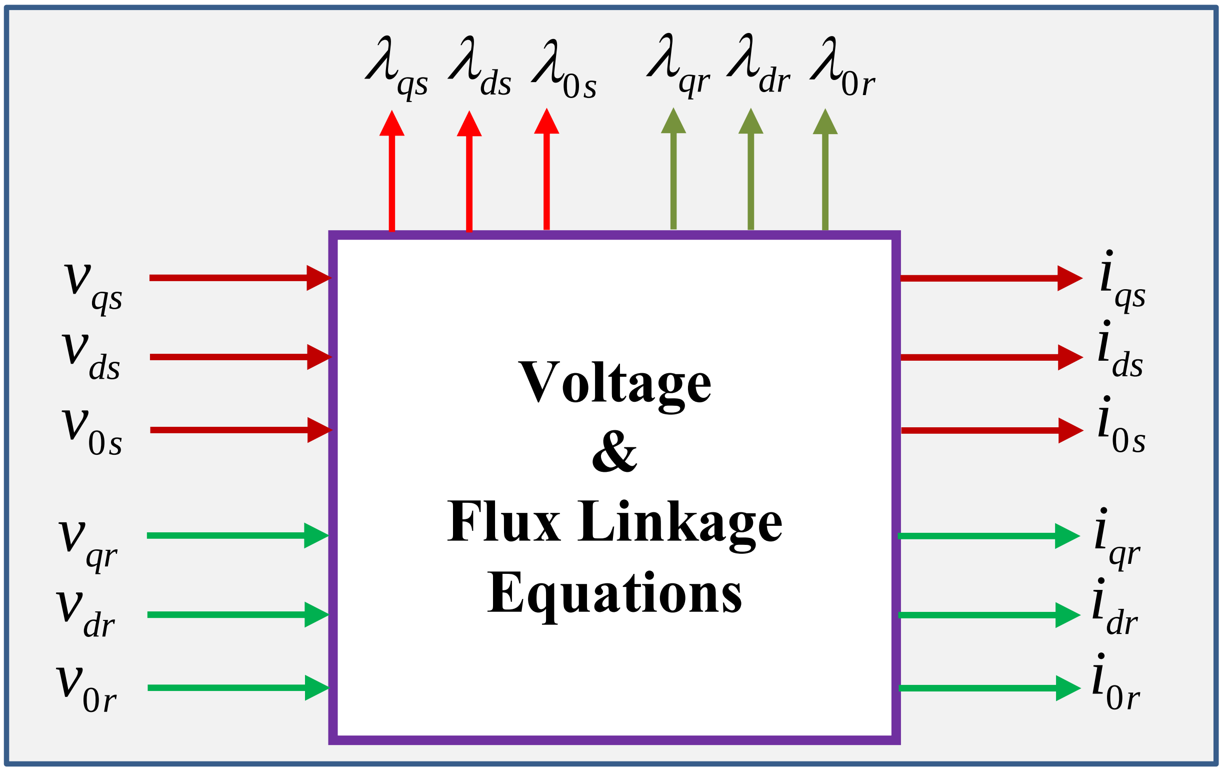
$$\left[\mathbf{f}_{qd0}\right]=\left[\mathbf{T}_{qd0}\right]\left[\mathbf{f}_{abc}\right]$$

Rotor voltages



Simulation of Induction Machines

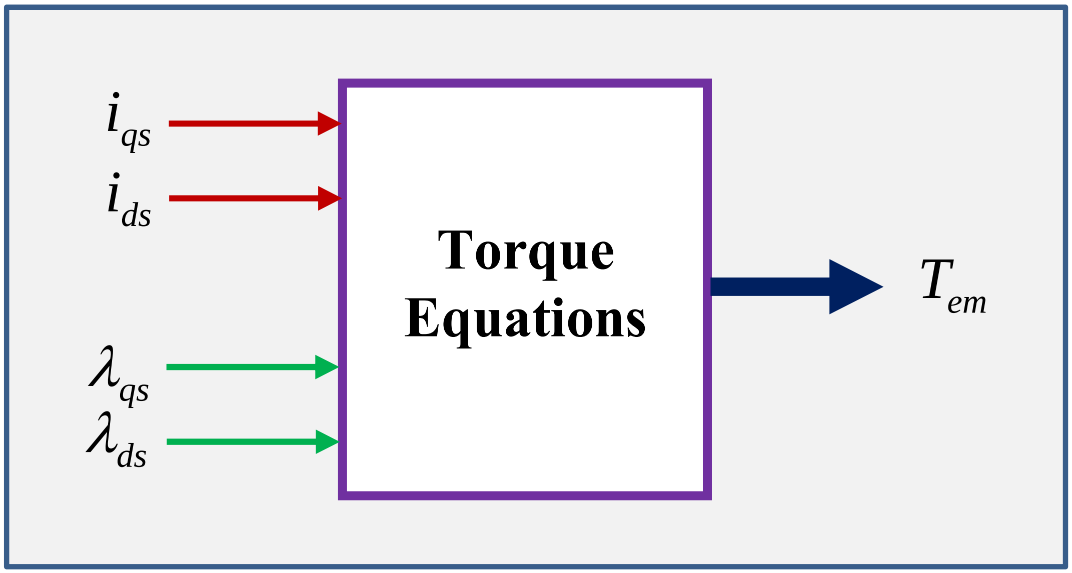
Sub-systems: Voltage & flux linkage equations



The stator and rotor equations are coupled.

Simulation of Induction Machines

Sub-systems: Torque equation

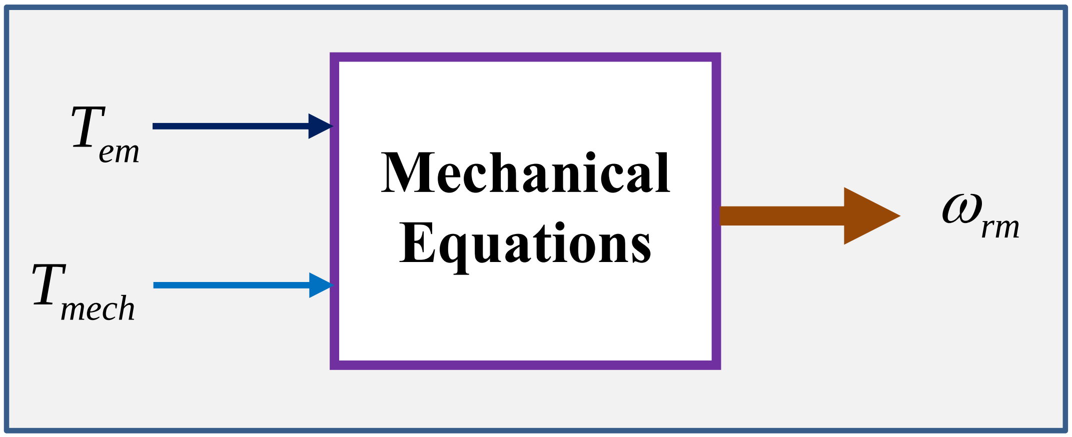


$$T_{em} = \frac{3}{2}\frac{P}{2}\left(\lambda_{ds} i_{qs} - \lambda_{qs} i_{ds}\right)$$

Note that rotor quantities or a combination of rotor and stator quantities can be used for electromagnetic torque calculation.

Simulation of Induction Machines

Sub-systems: Mechanical equation



Motoring Mode

$$T_{em} - T_{mech} - T_{damp} = J\frac{d\omega_{rm}}{dt}$$

Generating Mode

$$T_{em} + T_{mech} - T_{damp} = J\frac{d\omega_{rm}}{dt}$$

Simulation of Induction Machines

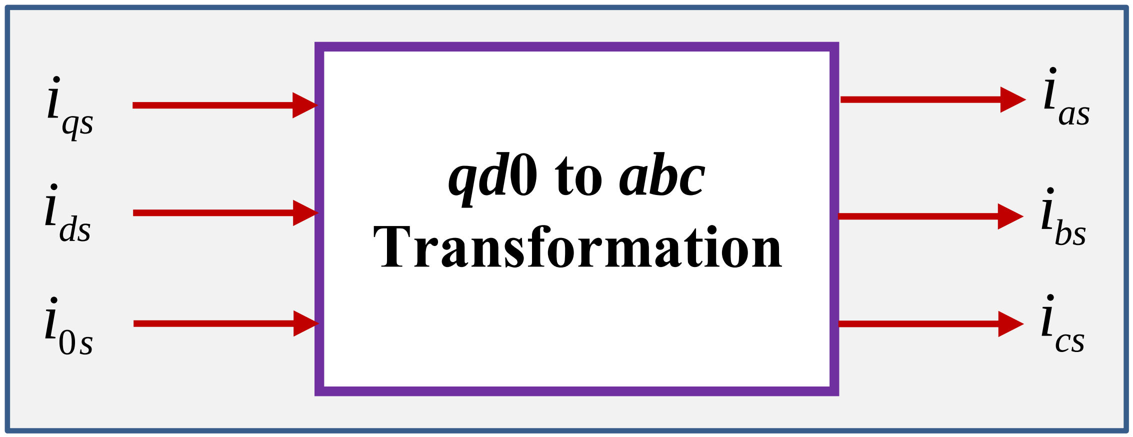
Sub-systems: Transformation from *qd*0 to *abc*

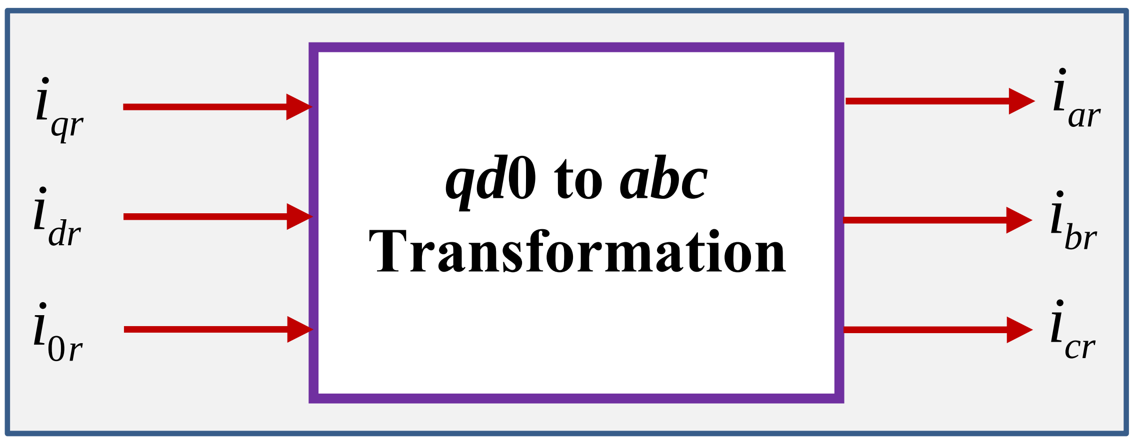
Stator Currents

$$[\mathbf{f}_{abc}] = [\mathbf{T}_{qd0}]^{-1} [\mathbf{f}_{qd0}]$$

Rotor Currents

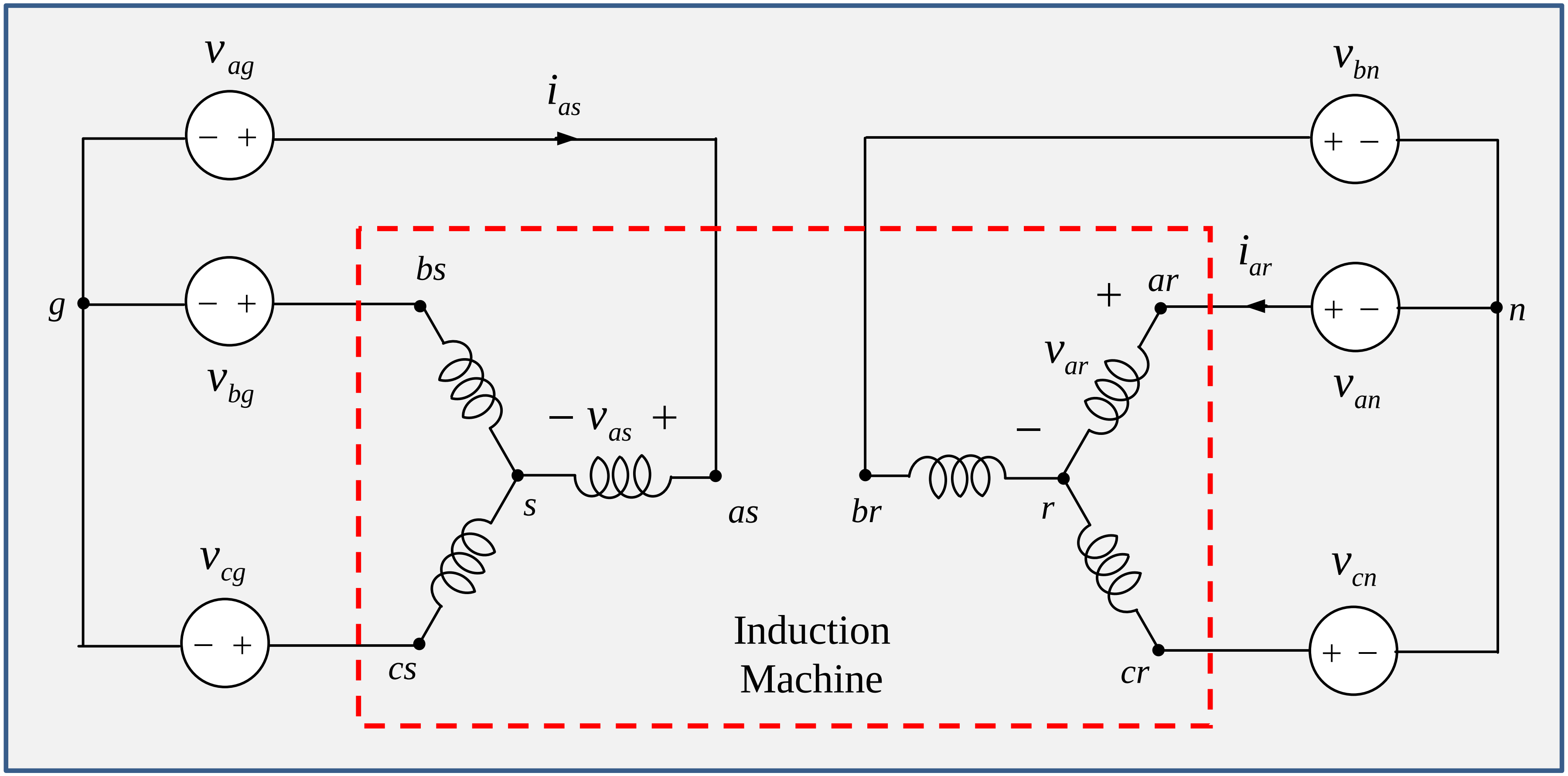
This transformation can also be used for flux linkages.





Simulation of Induction Machines in the Stationary RF

Voltage sources to windings connections



Simulation of Induction Machines in the Stationary RF

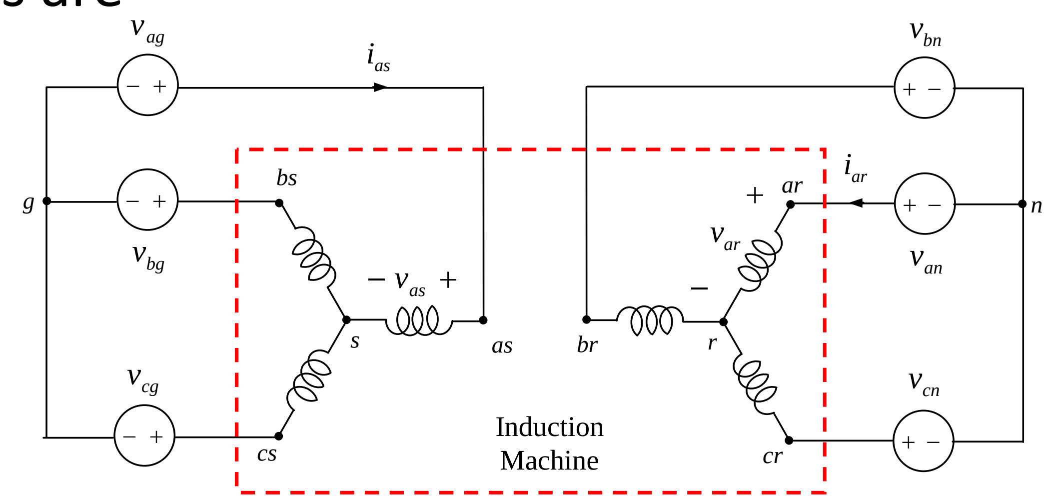
3-phase *P*-pole symmetrical IM

- The 3 applied voltages to the stator terminals need not be balanced nor sinusoidal.
- The 3 stator phase voltages are

$$v_{as} = v_{ag} - v_{sg}$$

$$v_{bs} = v_{bg} - v_{sg}$$

$$v_{cs} = v_{cg} - v_{sg}$$



- Or $3v_{sg} = (v_{ag} + v_{bg} + v_{cg}) - (v_{as} + v_{bs} + v_{cs})$

Simulation of Induction Machines in the Stationary RF

3-phase *P*-pole symmetrical IM

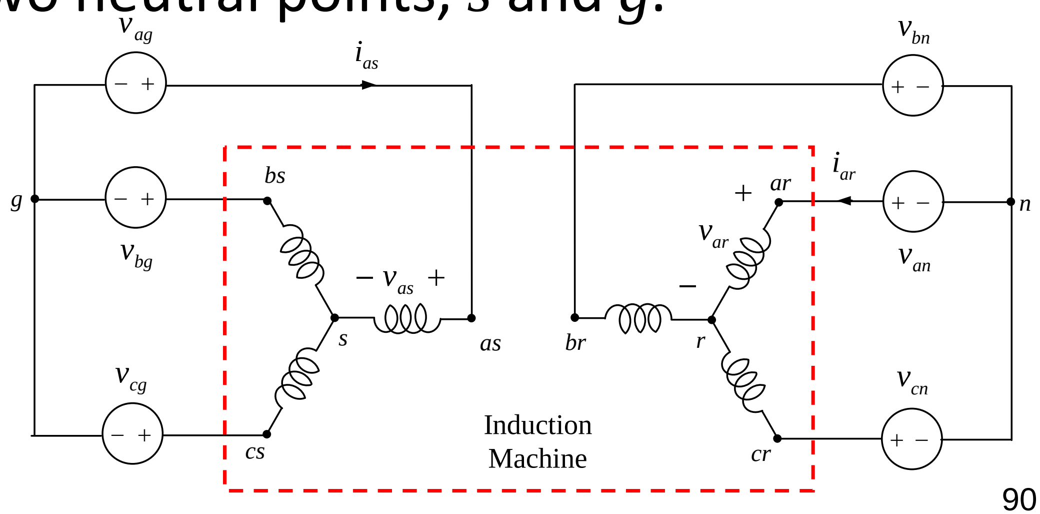
- v_{sg} can be determined as

$$v_{sg} = R_{sg}(i_{as} + i_{bs} + i_{cs}) + L_{sg}\tfrac{d}{dt}(i_{as} + i_{bs} + i_{cs}) = 3\left(R_{sg} + L_{sg}\tfrac{d}{dt}\right) i_{0s}$$

where R_{sg} and L_{sg} are the resistance and inductance of the connection between the two neutral points, s and g.

- Clearly when s and g are connected directly $v_{sg} = 0$.

- In the above expression if i_{0s}= 0 then v_{sg} = 0.



Simulation of Induction Machines in the Stationary RF

3-phase *P*-pole symmetrical IM

- In the case of **4-wire connection** (neutral points are connected together), i_{0s} is **zero** if a symmetrical induction machine is supplied by a set of applied voltages that are sinusoidal and balanced.

- In the case of **3-wire connection**, i_{0s} is always **zero** by physical constraint, irrespective of whether the 3-phase currents are balanced or not.

- In the case of **3-wire connection**, although i_{0s} is always **zero**, v_{sg} may **not be zero** depending on whether the applied voltages are sinusoidal and balanced.

Simulation of Induction Machines in the Stationary RF

3-phase *P*-pole symmetrical IM

- When the applied voltages are **non-sinusoidal** (such as the voltages of a six-step inverter), the **zero-sequence component of the applied voltage** may **not be zero**.

- In the case of **3-wire connection**, v_{sg} can be determined, in simulation, using the following expression with L_{sg} =0 and R_{sg} set to a high value to approximate the open-circuit condition.

$$v_{sg} = R_{sg}(i_{as} + i_{bs} + i_{cs}) + L_{sg}\tfrac{d}{dt}(i_{as} + i_{bs} + i_{cs}) = 3\left(R_{sg} + L_{sg}\tfrac{d}{dt}\right) i_{0s}$$

High value (for R_{sg})

0 (for L_{sg})

Simulation of Induction Machines in the Stationary RF

Transformation (Stator voltage)

- Transformation of the **stator voltages** from stator *abc* system to the **stationary** *qd*0 RF where *q*-axis is **aligned** with the stator *a*-axis.

$$[\mathbf{T}_{qd0}] = \frac{2}{3}\begin{bmatrix} \cos\theta & \cos\left(\theta - \frac{2\pi}{3}\right) & \cos\left(\theta + \frac{2\pi}{3}\right) \\ \sin\theta & \sin\left(\theta - \frac{2\pi}{3}\right) & \sin\left(\theta + \frac{2\pi}{3}\right) \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad \xrightarrow{\theta = 0} \quad [\boldsymbol{T}_{qd0}] = \frac{2}{3}\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$[\mathbf{v}_{qd0}] = [\mathbf{T}_{qd0}]\,[\mathbf{v}_{abc}]$$

$$v_{qs}^{s} = \tfrac{2}{3} v_{as} - \tfrac{1}{3} v_{bs} - \tfrac{1}{3} v_{cs} = \tfrac{2}{3} v_{ag} - \tfrac{1}{3} v_{bg} - \tfrac{1}{3} v_{cg}$$

$$v_{ds}^{s} = \tfrac{-1}{\sqrt{3}}(v_{bs} - v_{cs}) = \tfrac{-1}{\sqrt{3}}(v_{bg} - v_{cg})$$

$$v_{0s}^{s} = \tfrac{1}{3}(v_{as} + v_{bs} + v_{cs}) = \tfrac{1}{3}(v_{ag} + v_{bg} + v_{cg}) - v_{sg}$$

Simulation of Induction Machines in the Stationary RF

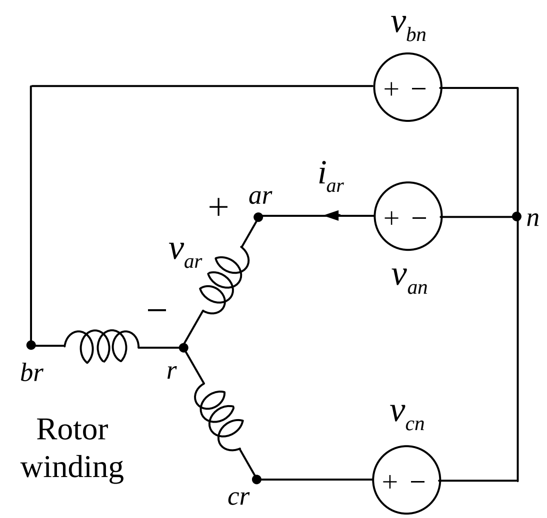
Transformation (rotor voltage; 1st transformation)

- Transformation of the rotor voltages **referred to stator side** from rotor *abc* system to the *qd*0 RF **fixed to rotor** where *q*-axis is **aligned** with the rotor *a*-axis.

$$v'^{r}_{qr} = \frac{2}{3}v'_{ar} - \frac{1}{3}v'_{br} - \frac{1}{3}v'_{cr} = \frac{2}{3}v'_{an} - \frac{1}{3}v'_{bn} - \frac{1}{3}v'_{cn}$$

$$v'^{r}_{dr} = \frac{-1}{\sqrt{3}}(v'_{br} - v'_{cr}) = \frac{-1}{\sqrt{3}}(v'_{bn} - v'_{cn})$$

$$v'^{r}_{0r} = \frac{1}{3}(v'_{ar} + v'_{br} + v'_{cr}) = \frac{1}{3}(v'_{an} + v'_{bn} + v'_{cn}) - v'_{rn}$$



where v'_{rn} is the voltage between points *r* and *n*. The prime denotes values referred to the stator side.

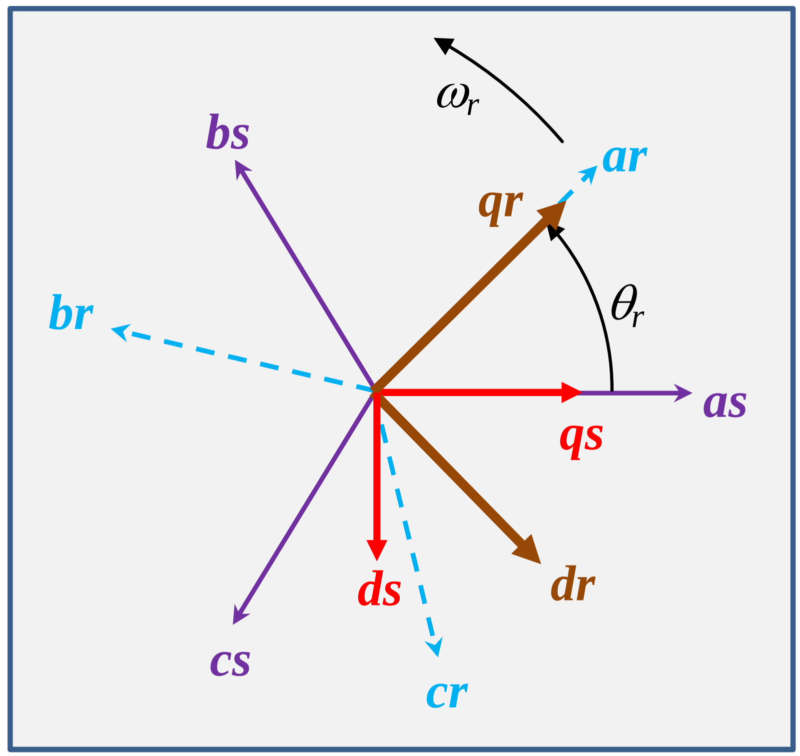
Simulation of Induction Machines in the Stationary RF

Transformation (rotor voltage; 2nd transformation)

- Transformation of the rotor voltages **referred to stator side** from the $qd0$ RF **fixed to rotor** to the **stationary** $qd0$ RF.

$$v'^{s}_{qr} = v'^{r}_{qr} \cos\theta_r + v'^{r}_{dr} \sin\theta_r$$

$$v'^{s}_{dr} = -v'^{r}_{qr} \sin\theta_r + v'^{r}_{dr} \cos\theta_r$$



where the rotor angle is

$$\theta_r = \int_0^t \omega_r \, dt + \theta_r(0)$$

Simulation of Induction Machines in the Stationary RF

q-axis flux linkage equations

Stator

$$\psi_{qs}^{s} = x_{ls}\, i_{qs}^{s} + \psi_{mq}^{s} \;\Rightarrow\; i_{qs}^{s} = \frac{\psi_{qs}^{s} - \psi_{mq}^{s}}{x_{ls}} \quad (1)$$

Magnetizing

$$\psi_{mq}^{s} = x_{m}\left(i_{qs}^{s} + i_{qr}^{\prime s}\right)$$

Rotor

$$\psi_{qr}^{\prime s} = x_{lr}^{\prime}\, i_{qr}^{\prime s} + \psi_{mq}^{s} \;\Rightarrow\; i_{qr}^{\prime s} = \frac{\psi_{qr}^{\prime s} - \psi_{mq}^{s}}{x_{lr}^{\prime}} \quad (2)$$

$$\psi_{mq}^{s} = x_{M}\left(\frac{\psi_{qs}^{s}}{x_{ls}} + \frac{\psi_{qr}^{\prime s}}{x_{lr}^{\prime}}\right) \quad (3)$$

where

$$\frac{1}{x_{M}} = \frac{1}{x_{m}} + \frac{1}{x_{ls}} + \frac{1}{x_{lr}^{\prime}}$$

Simulation of Induction Machines in the Stationary RF

d-axis flux linkage equations

Stator

$$\psi_{ds}^{s} = x_{ls}\, i_{ds}^{s} + \psi_{md}^{s} \;\Rightarrow\; i_{ds}^{s} = \frac{\psi_{ds}^{s} - \psi_{md}^{s}}{x_{ls}} \quad (6)$$

Magnetizing

$$\psi_{md}^{s} = x_{m}\left(i_{ds}^{s} + i_{dr}^{\prime s}\right)$$

Rotor

$$\psi_{dr}^{\prime s} = x_{lr}^{\prime}\, i_{dr}^{\prime s} + \psi_{md}^{s} \;\Rightarrow\; i_{dr}^{\prime s} = \frac{\psi_{dr}^{\prime s} - \psi_{md}^{s}}{x_{lr}^{\prime}} \quad (7)$$

$$\psi_{md}^{s} = x_{M}\left(\frac{\psi_{ds}^{s}}{x_{ls}} + \frac{\psi_{dr}^{\prime s}}{x_{lr}^{\prime}}\right) \quad (8)$$

where

$$\frac{1}{x_M} = \frac{1}{x_m} + \frac{1}{x_{ls}} + \frac{1}{x_{lr}^{\prime}}$$

Simulation of Induction Machines in the Stationary RF

Stator voltage equations

q-axis

4

$$v_{qs}^{s} = r_s\, i_{qs}^{s} + \tfrac{1}{\omega_b}\frac{d\psi_{qs}^{s}}{dt} \quad \& \quad i_{qs}^{s} = \frac{\psi_{qs}^{s} - \psi_{mq}^{s}}{x_{ls}} \Rightarrow \psi_{qs}^{s} = \omega_b \int \left\{ v_{qs}^{s} + \tfrac{r_s}{x_{ls}}\left(\psi_{mq}^{s} - \psi_{qs}^{s}\right) \right\} dt$$

d-axis

9

$$v_{ds}^{s} = r_s\, i_{ds}^{s} + \tfrac{1}{\omega_b}\frac{d\psi_{ds}^{s}}{dt} \quad \& \quad i_{ds}^{s} = \frac{\psi_{ds}^{s} - \psi_{md}^{s}}{x_{ls}} \Rightarrow \psi_{ds}^{s} = \omega_b \int \left\{ v_{ds}^{s} + \tfrac{r_s}{x_{ls}}\left(\psi_{md}^{s} - \psi_{ds}^{s}\right) \right\} dt$$

Zero

$$v_{0s} = r_s\, i_{0s} + \tfrac{1}{\omega_b}\frac{d\psi_{0s}}{dt} \quad \& \quad \psi_{0s} = x_{ls}\, i_{0s} \Rightarrow i_{0s} = \tfrac{\omega_b}{x_{ls}} \int \left(v_{0s} - r_s\, i_{0s}\right) dt$$

Simulation of Induction Machines in the Stationary RF

Rotor voltage equations

$$v'^{s}_{qr} = r'_r i'^{s}_{qr} + \frac{1}{\omega_b}\frac{d\psi'^{s}_{qr}}{dt} - \frac{\omega_r}{\omega_b}\psi'^{s}_{dr}$$

$$i'^{s}_{qr} = \frac{\psi'^{s}_{qr} - \psi^{s}_{mq}}{x'_{lr}}$$

q-axis →

$$\psi'^{s}_{qr} = \omega_b \int \left\{ v'^{s}_{qr} + \frac{\omega_r}{\omega_b}\psi'^{s}_{dr} + \frac{r'_r}{x'_{lr}}\left(\psi^{s}_{mq} - \psi'^{s}_{qr}\right) \right\} dt$$

5

$$v'^{s}_{dr} = r'_r i'^{s}_{dr} + \frac{1}{\omega_b}\frac{d\psi'^{s}_{dr}}{dt} + \frac{\omega_r}{\omega_b}\psi'^{s}_{qr}$$

$$i'^{s}_{dr} = \frac{\psi'^{s}_{dr} - \psi^{s}_{md}}{x'_{lr}}$$

d-axis →

$$\psi'^{s}_{dr} = \omega_b \int \left\{ v'^{s}_{dr} - \frac{\omega_r}{\omega_b}\psi'^{s}_{qr} + \frac{r'_r}{x'_{lr}}\left(\psi^{s}_{md} - \psi'^{s}_{dr}\right) \right\} dt$$

10

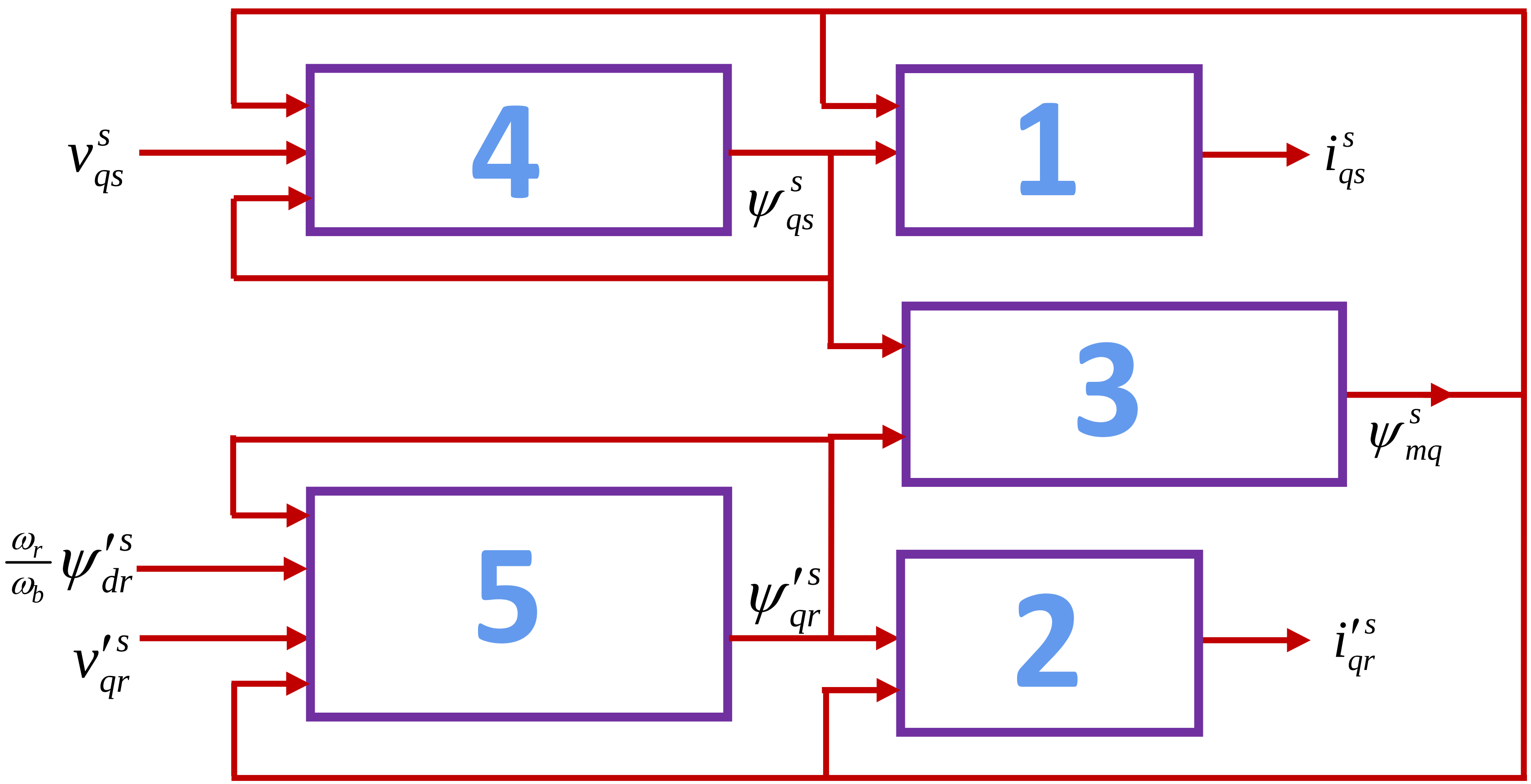
$$v'_{0r} = r'_r i'_{0r} + \frac{1}{\omega_b}\frac{d\psi'_{0r}}{dt}$$ & $$\psi'_{0r} = x'_{lr} i'_{0r}$$

Zero →

$$i'_{0r} = \frac{\omega_b}{x'_{lr}} \int \left(v'_{0r} - r'_r i'_{0r} \right) dt$$

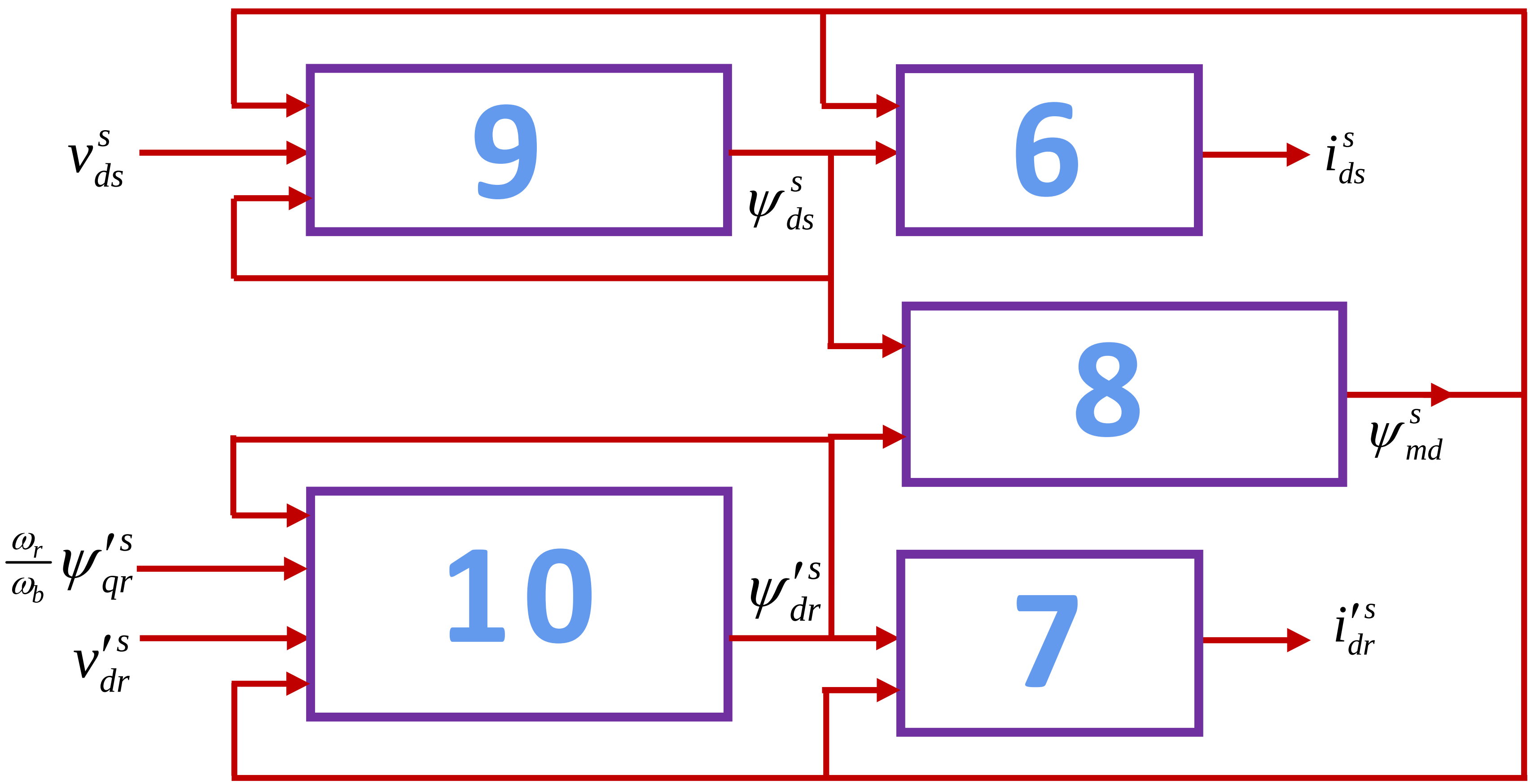
Simulation of Induction Machines in the Stationary RF

q-axis voltage and flux linkage block diagram



Simulation of Induction Machines in the Stationary RF

d-axis voltage and flux linkage block diagram



Simulation of Induction Machines in the Stationary RF

Torque and Mechanical equations

- The torque equation is

$$T_{em} = \frac{3}{2}\frac{P}{2\omega_b}\left(\psi_{ds}i_{qs} - \psi_{qs}i_{ds}\right)$$

- The dynamic mechanical equation is

Motoring Mode

$$T_{em} - T_{mech} - T_{damp} = J\frac{d\omega_{rm}}{dt}$$

Generating Mode

$$T_{em} + T_{mech} - T_{damp} = J\frac{d\omega_{rm}}{dt}$$

Simulation of Induction Machines in the Stationary RF

Transformation (Stator currents)

- Transformation of the stator currents from the stationary $qd0$ RF to the stator abc system.

$$[\mathbf{i}_{abc}] = [\mathbf{T}_{qd0}]^{-1}[\mathbf{i}_{qd0}]$$

where

$$[\mathbf{T}_{qd0}]^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 1 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 1 \end{bmatrix}$$

$$\begin{cases} i_{as} = i_{qs}^{s} + i_{0s} \\ i_{bs} = -\frac{1}{2} i_{qs}^{s} - \frac{\sqrt{3}}{2} i_{ds}^{s} + i_{0s} \\ i_{cs} = -\frac{1}{2} i_{qs}^{s} + \frac{\sqrt{3}}{2} i_{ds}^{s} + i_{0s} \end{cases}$$

Simulation of Induction Machines in the Stationary RF

Transformation (rotor currents; 1st transformation)

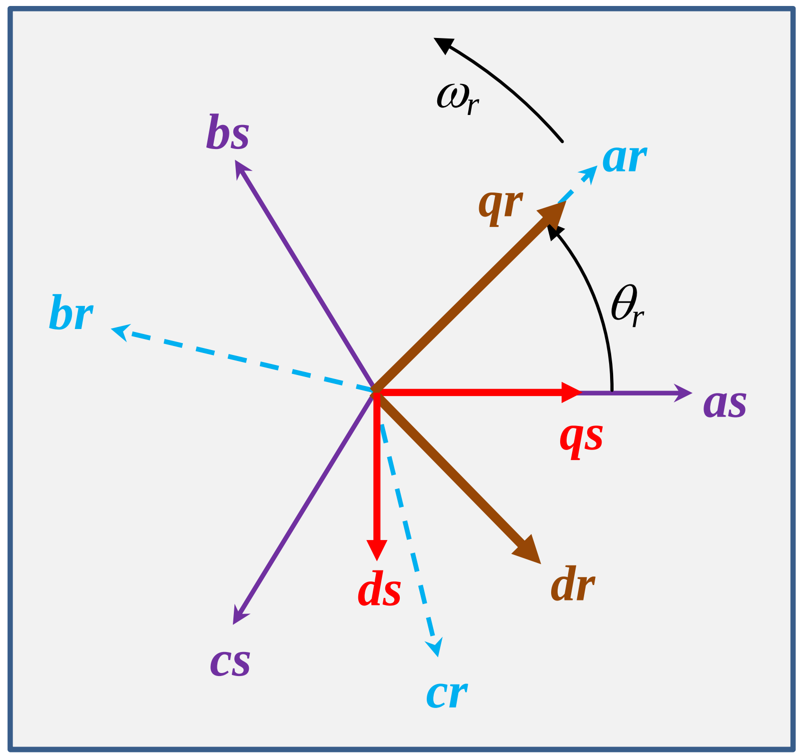
- Transformation of the rotor currents **referred to stator side** from the stationary $qd0$ RF to the $qd0$ RF fixed to rotor .

$$i'^{r}_{qr} = i'^{s}_{qr}\cos\theta_r - i'^{s}_{dr}\sin\theta_r$$

$$i'^{r}_{dr} = i'^{s}_{qr}\sin\theta_r + i'^{s}_{dr}\cos\theta_r$$

where the rotor angle is

$$\theta_r = \int_0^t \omega_r\, dt + \theta_r(0)$$



Simulation of Induction Machines in the Stationary RF

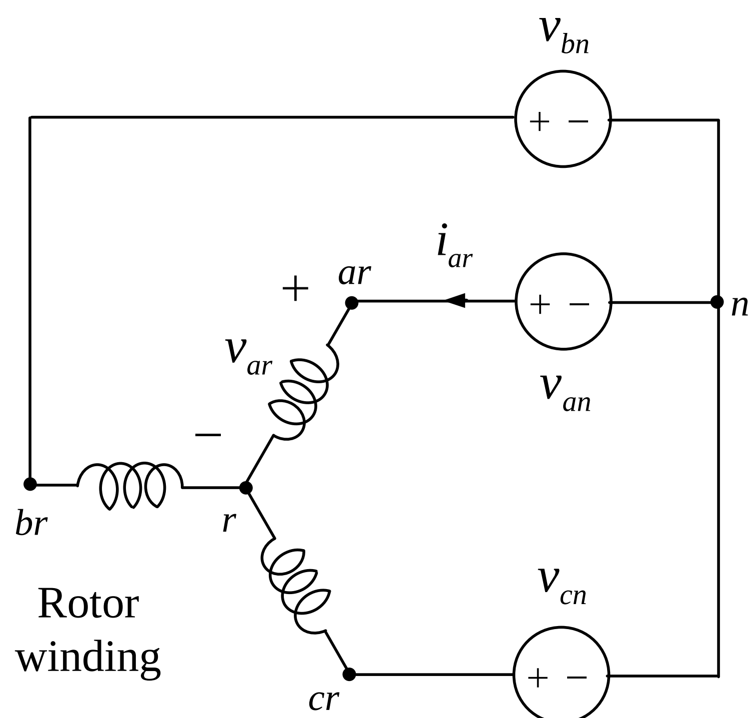
Transformation (rotor currents; 2nd transformation)

- Transformation of the rotor currents **referred to stator side** from the $qd0$ RF **fixed to rotor** to the rotor abc system.

$$i'_{ar} = i'^{r}_{qr} + i'_{0r}$$

$$i'_{br} = -\frac{1}{2} i'^{r}_{qr} - \frac{\sqrt{3}}{2} i'^{r}_{dr} + i'_{0r}$$

$$i'_{cr} = -\frac{1}{2} i'^{r}_{qr} + \frac{\sqrt{3}}{2} i'^{r}_{dr} + i'_{0r}$$



Per-Unit System

Base quantities

- The base quantities with **peak** rather than rms value of a P-pole, three-phase induction machine with **rated line-to-line rms voltage**, V_{rated}, and **rated volt-ampere**, S_{rated}, are as follows:

Base Voltage $$V_b = \sqrt{\tfrac{2}{3}}\, V_{rated}$$

Base Volt-ampere $$S_b = S_{rated}$$

Base Peak Current $$I_b = 2S_b/(3V_b)$$

Base Impedance $$Z_b = V_b / I_b$$

Base Torque $$T_b = S_b / \omega_{bm}$$ where $$\omega_{bm} = 2\omega_b / P$$

Per-Unit System

Torque Equation

$$T_{em} = \frac{3}{2}\frac{P}{2\omega_b}\left(\psi_{ds} i_{qs} - \psi_{qs} i_{ds}\right)$$

$$T_b = \frac{S_b}{\omega_{bm}}$$

$\omega_{bm} = 2\omega_b / P$ →

$$T_b = \frac{S_b}{\frac{2}{P}\omega_b}$$

$I_b = 2S_b/(3V_b)$ →

$$T_b = \frac{3}{2}\frac{P}{2}\frac{V_b I_b}{\omega_b}$$

$$\frac{T_{em}}{T_b} = \frac{\frac{3}{2}\frac{P}{2\omega_b}\left(\psi_{ds} i_{qs} - \psi_{qs} i_{ds}\right)}{\frac{3}{2}\frac{P}{2\omega_b} V_b I_b}$$

→

$$T_{em}(pu) = \psi_{ds}(pu) i_{qs}(pu) - \psi_{qs}(pu) i_{ds}(pu)$$

where

$$\psi_{ds}(pu) = \frac{\psi_{ds}}{V_b} \qquad \psi_{qs}(pu) = \frac{\psi_{qs}}{V_b} \qquad i_{qs}(pu) = \frac{i_{qs}}{I_b} \qquad i_{ds}(pu) = \frac{i_{ds}}{I_b}$$

Per-Unit System

Mechanical Equation

$$J\frac{d\omega_{rm}}{dt} = T_{em} + T_{mech} - T_{damp}$$

$$\omega_{rm} = 2\omega_r / P$$

$$\frac{2J\omega_b}{P}\frac{d(\omega_r/\omega_b)}{dt} = T_{em} + T_{mech} - T_{damp} \quad *$$

*

$$T_b = S_b / \omega_{bm}$$

$$\omega_{bm} = 2\omega_b / P$$

$$2H\frac{d(\omega_r/\omega_b)}{dt} = T_{em}(pu) + T_{mech}(pu) - T_{damp}(pu)$$

where $H = J\omega_{bm}^2/(2S_b)$ is the inertia constant.

Machine Parameters

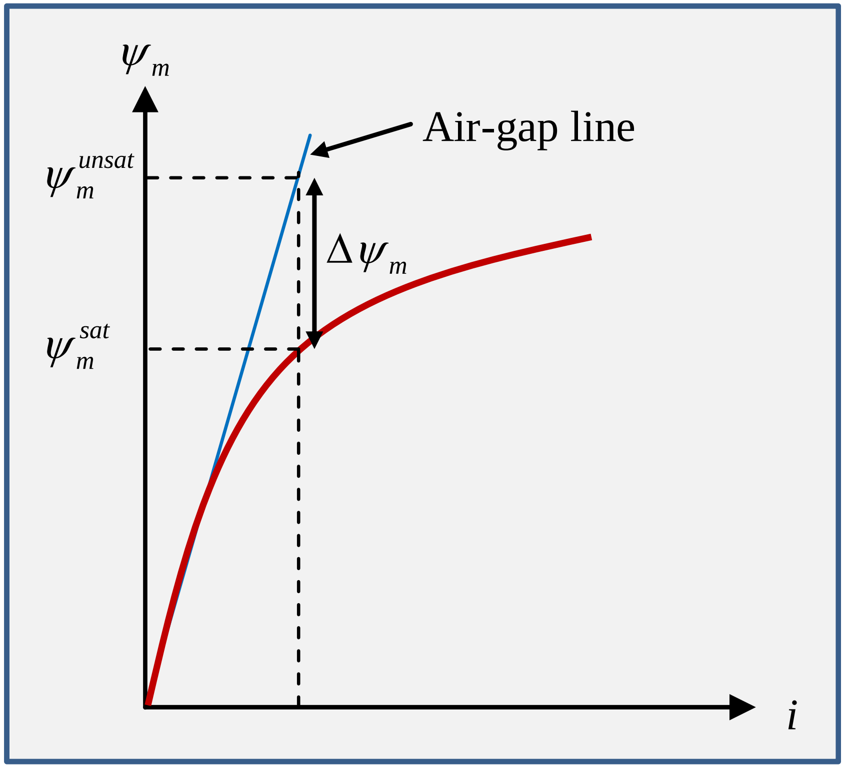
The following parameters/quantities are required for the simulation

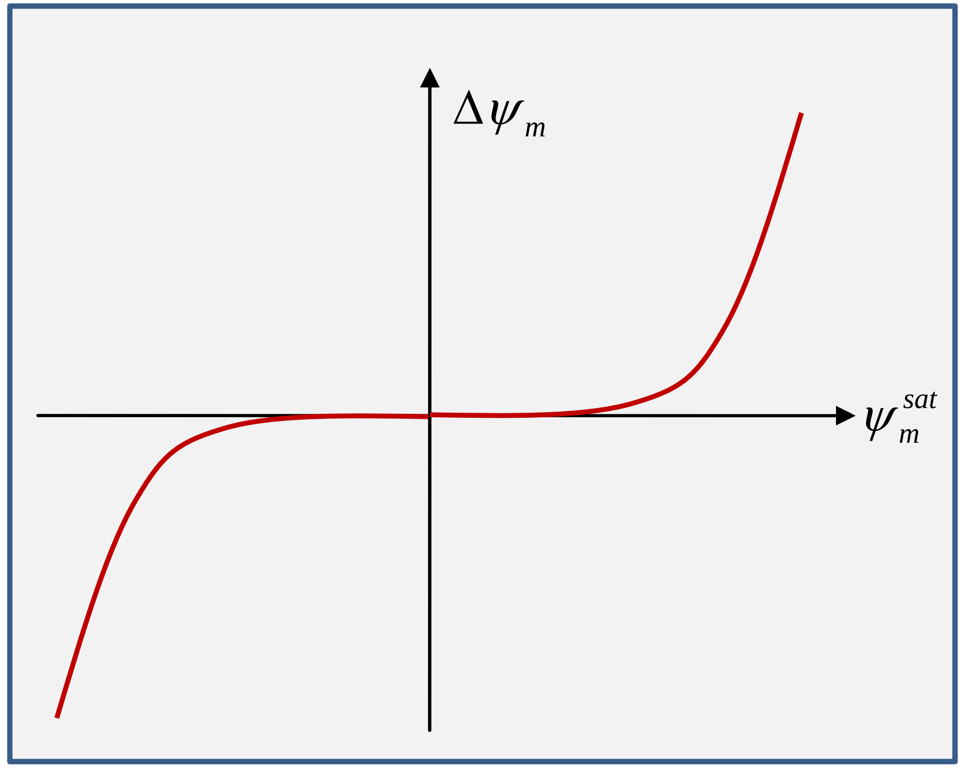
- x_{ls} stator or armature winding **leakage** reactance
- x'_{lr} rotor winding **leakage** reactance referred to stator
- x_m **magnetizing** reactance
- r_s stator or armature winding **resistance**
- r'_r rotor winding **resistance** referred to stator
- J rotor **moment of inertia**
- P **number of poles**
- ω_b rotor **base speed**
- v_{rated} rated line-to-line rms **voltage of stator**
- S_{rared} rated **volt-ampere**
- T_{mech} mechanical **load torque** profile

Saturation of Mutual Flux

- The iron saturation mainly affects the value of the **magnetizing inductance** and, to a much lesser extent, the **leakage inductances**.
- Considering the iron saturation effects on the **leakage inductances** is complex (because of the complicated leakage flux path) and these effects are **neglected** here.
- Therefore the iron saturation effects on the **magnetizing inductance** are only **considered**.
- We assume that the iron saturation affects the *q*- and *d*-axis components in the **same manner** (due to smooth air-gap).

Saturation of Mutual Flux





Saturation Characteristics

Saturation of Mutual Flux

- The saturated value of the mutual flux linkage per second in the *q*-axis is given by:

$$\psi_{mq}^{s,sat} = \psi_{mq}^{s,unsat} - \Delta\psi_{mq}^{s}$$

$$\psi_{mq}^{s,sat} = x_m \left(\frac{\psi_{qs}^{s} - \psi_{mq}^{s,sat}}{x_{ls}} + \frac{\psi_{qr}'^{s} - \psi_{mq}^{s,sat}}{x_{lr}'} \right) - \Delta\psi_{mq}^{s}$$

- Rearranging subject to $\psi_{mq}^{s,sat}$ yields

$$\psi_{mq}^{s,sat} = x_M \left(\frac{\psi_{qs}^{s}}{x_{ls}} + \frac{\psi_{qr}'^{s}}{x_{lr}'} - \frac{\Delta\psi_{mq}^{s}}{x_m} \right)$$

where

Unsaturated

$$\frac{1}{x_M} = \frac{1}{x_m} + \frac{1}{x_{ls}} + \frac{1}{x_{lr}'}$$

Saturation of Mutual Flux

- Similarly the saturated value of the mutual flux linkage per second in the *d*-axis is given by:

$$\psi_{md}^{s,sat} = \psi_{md}^{s,unsat} - \Delta\psi_{md}^{s}$$

$$\psi_{md}^{s,sat} = x_m \left(\frac{\psi_{ds}^{s} - \psi_{md}^{s,sat}}{x_{ls}} + \frac{\psi_{dr}'^{s} - \psi_{md}^{s,sat}}{x_{lr}'} \right) - \Delta\psi_{md}^{s}$$

- Rearranging subject to $\psi_{md}^{s,sat}$ yields

$$\psi_{md}^{s,sat} = x_M \left(\frac{\psi_{ds}^{s}}{x_{ls}} + \frac{\psi_{dr}'^{s}}{x_{lr}'} - \frac{\Delta\psi_{md}^{s}}{x_m} \right)$$

where

Unsaturated

$$\frac{1}{x_M} = \frac{1}{x_m} + \frac{1}{x_{ls}} + \frac{1}{x_{lr}'}$$

Saturation of Mutual Flux

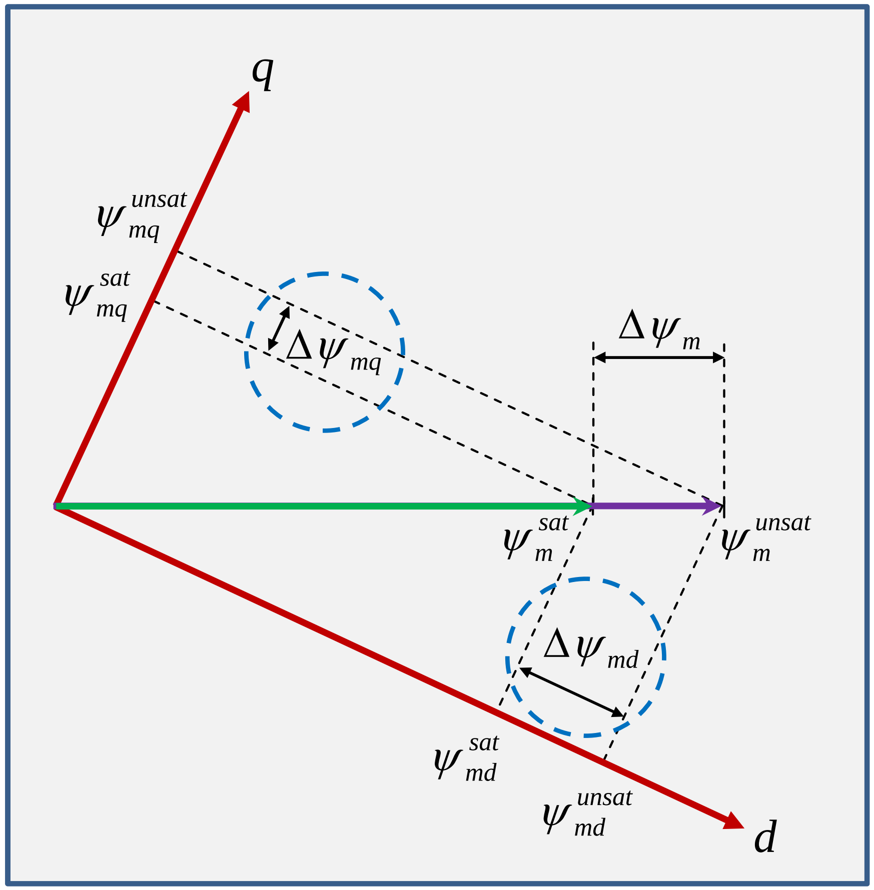
- Assuming a **proportional reduction** in flux linkages of the *q*- and *d*-axis yields:

$$\Delta\psi_{mq}^{s} = \frac{\psi_{mq}^{s,sat}}{\psi_{m}^{sat}}\Delta\psi_{m}$$

$$\Delta\psi_{md}^{s} = \frac{\psi_{md}^{s,sat}}{\psi_{m}^{sat}}\Delta\psi_{m}$$

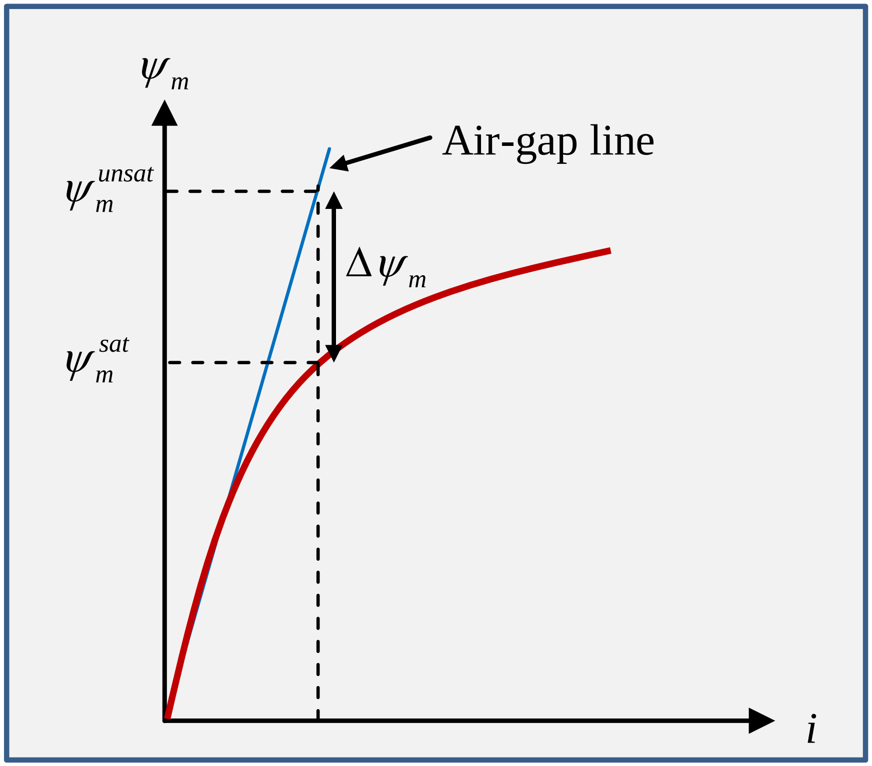
where

$$\psi_{m}^{sat} = \sqrt{\left(\psi_{mq}^{s,sat}\right)^2 + \left(\psi_{md}^{s,sat}\right)^2}$$

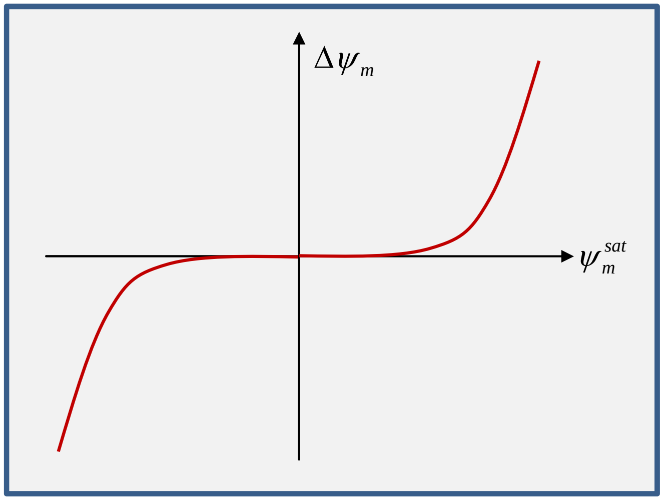


Saturation of Mutual Flux

- The relationship between $\Delta\psi_m$ and ψ_m^{sat} can be determined from the no-load test curve of the machine.

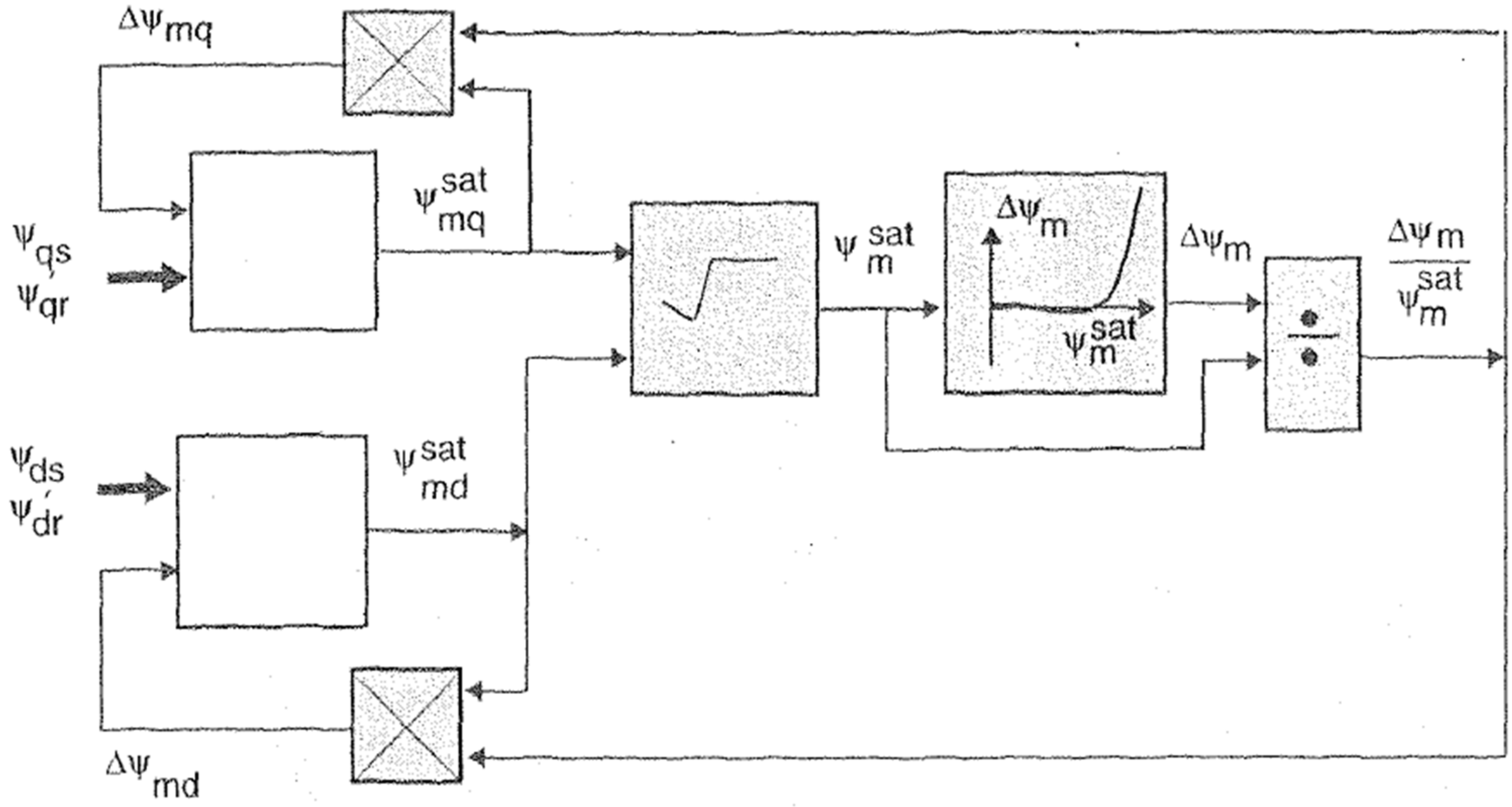


- The value of $\Delta\psi_m$ is obtained from ψ_m^{sat} using piece-wise segments or a look-up table as explained for the transformers.



Saturation of Mutual Flux

- The part of the IM simulation that is **affected** by the **inclusion of mutual flux saturation** is shown below:



Linearization of Induction Machine Model

- For nonlinear models, the well-established linear control techniques **cannot** be employed.

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \\ \mathbf{y} = \mathbf{g}(\mathbf{x}, \mathbf{u}) \end{cases}$$

→ **Linear control theory cannot be used.**

- If the nonlinear model can be linearized around an equilibrium point, then the linear control techniques **can** be used.

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{cases}$$

→ **Linear control theory can be used.**

Linearization of Induction Machine Model

When can a nonlinear model be linearized?

1. If the system is always working around an **equilibrium point**.

   Or

2. If the nonlinear terms are **not significant** and can be approximated by linear terms.

The linearization methods

1. **Taylor** series expansion
2. **Perturbation** method

Linearization of Induction Machine Model

Nonlinear IM model in the arbitrary qd0 RF

$$v_{qs} = r_s\, i_{qs} + \frac{1}{\omega_b}\frac{d\psi_{qs}}{dt} + \frac{\omega}{\omega_b}\psi_{ds}$$

$$v_{ds} = r_s\, i_{ds} + \frac{1}{\omega_b}\frac{d\psi_{ds}}{dt} - \frac{\omega}{\omega_b}\psi_{qs}$$

$$v'_{qr} = r'_r\, i'_{qr} + \frac{1}{\omega_b}\frac{d\psi'_{qr}}{dt} + \frac{\omega-\omega_r}{\omega_b}\psi'_{dr}$$

$$v'_{dr} = r'_r\, i'_{dr} + \frac{1}{\omega_b}\frac{d\psi'_{dr}}{dt} - \frac{\omega-\omega_r}{\omega_b}\psi'_{qr}$$

$$T_{em} - T_{mech} - T_{damp} = J\frac{d\omega_{rm}}{dt}$$

where $T_{em} = \frac{3}{2}\frac{P}{2\omega_b}\left(\psi_{ds} i_{qs} - \psi_{qs} i_{ds}\right)$

Nonlinear terms

In the case of stationary RF:

Linearization of Induction Machine Model

Nonlinear IM model in the synchronous qd0 RF

$$\begin{bmatrix} v_{qs}^e \\ v_{ds}^e \\ v_{qr}'^e \\ v_{dr}'^e \end{bmatrix} = \begin{bmatrix} r_s + \frac{p}{\omega_b} x_{ss} & \frac{\omega_e}{\omega_b} x_{ss} & \frac{p}{\omega_b} x_m & \frac{\omega_e}{\omega_b} x_m \\ -\frac{\omega_e}{\omega_b} x_{ss} & r_s + \frac{p}{\omega_b} x_{ss} & -\frac{\omega_e}{\omega_b} x_m & \frac{p}{\omega_b} x_m \\ \frac{p}{\omega_b} x_m & s\frac{\omega_e}{\omega_b} x_m & r_r' + \frac{p}{\omega_b} x_{rr}' & s\frac{\omega_e}{\omega_b} x_{rr}' \\ -s\frac{\omega_e}{\omega_b} x_m & \frac{p}{\omega_b} x_m & -s\frac{\omega_e}{\omega_b} x_{rr}' & r_r' + \frac{p}{\omega_b} x_{rr}' \end{bmatrix} \begin{bmatrix} i_{qs}^e \\ i_{ds}^e \\ i_{qr}'^e \\ i_{dr}'^e \end{bmatrix}$$

Nonlinear terms are encircled.

$$T_{em} - T_{mech} - T_{damp} = J\frac{d\omega_{rm}}{dt}$$

$$T_{em} = \frac{3}{2}\frac{P}{2\omega_b}\left(\psi_{ds} i_{qs} - \psi_{qs} i_{ds}\right)$$

where

$$x_{ss} = x_{ls} + x_m$$

$$x_{rr}' = x_{lr}' + x_m$$

$$s = \frac{\omega_e - \omega_r}{\omega_e}$$

First Technique:

Taylor Series Expansion

Linearization of Nonlinear Systems

Consider a system whose input is $x(t)$ and output is $y(t)$. The relationship between $y(t)$ and $x(t)$ is given by

$$y = f(x)$$

If the normal operating condition corresponds to x_o, y_o then above relation may be expanded into a Taylor series about this point as follows:

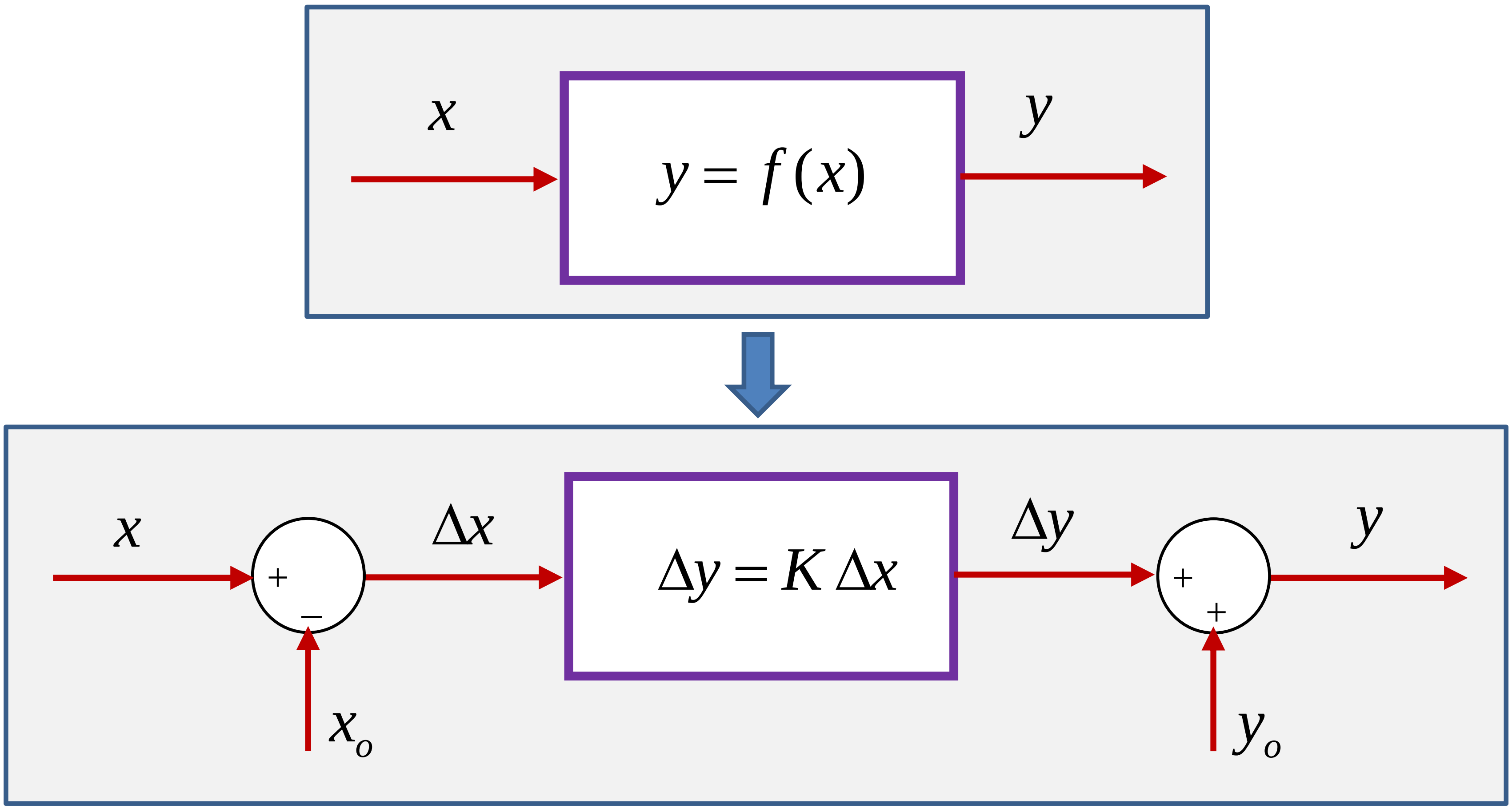
$$y = f(x_o) + \left.\frac{df}{dx}\right|_{x=x_o}(x - x_o) + \frac{1}{2!}\left.\frac{d^2 f}{dx^2}\right|_{x=x_o}(x - x_o)^2 + \cdots$$

Neglecting the higher-order terms yields $y = y_o + K(x - x_o)$

where $y_o = f(x_o)$ and $K = \left.\frac{df}{dx}\right|_{x=x_o}$

Finally assuming $\Delta x = x - x_o$ and $\Delta y = y - y_o$ yields $\Delta y = K\,\Delta x$

Linearization of Nonlinear Systems



$$\Delta x = x - x_o \qquad K = \left.\frac{df}{dx}\right|_{x=x_o} \qquad y = y_o + \Delta y$$

Linearization of Nonlinear Systems

Consider a system whose input is $x_1(t)$ and $x_2(t)$ and output is $y(t)$. The relationship between $y(t)$ and the inputs is given by

$$y = f\left(x_1, x_2\right)$$

Expanding Taylor series about equilibrium point x_{1o}, x_{2o}, y_o

$$y = f\left(x_{1o}, x_{2o}\right) + \left[\left.\frac{\partial f}{\partial x_1}\right|_{\substack{x_1 = x_{1o} \\ x_2 = x_{2o}}} \left(x_1 - x_{1o}\right) + \left.\frac{\partial f}{\partial x_2}\right|_{\substack{x_1 = x_{1o} \\ x_2 = x_{2o}}} \left(x_2 - x_{2o}\right) \right]$$
$$+ \frac{1}{2!}\left[\left.\frac{\partial^2 f}{\partial x_1^2}\right|_{\substack{x_1 = x_{1o} \\ x_2 = x_{2o}}} \left(x_1 - x_{1o}\right)^2 + \left.\frac{\partial^2 f}{\partial x_2^2}\right|_{\substack{x_1 = x_{1o} \\ x_2 = x_{2o}}} \left(x_2 - x_{2o}\right)^2 \right.$$
$$\left. + 2\left.\frac{\partial^2 f}{dx_1 \partial x_2}\right|_{\substack{x_1 = x_{1o} \\ x_2 = x_{2o}}} \left(x_1 - x_{1o}\right)\left(x_2 - x_{2o}\right) \right] + \cdots$$

Linearization of Nonlinear Systems

Neglecting the higher-order terms yields

$$y = f(x_1, x_2) \;\Rightarrow\; y = f(x_{1o}, x_{2o}) + \left[\left.\frac{\partial f}{\partial x_1}\right|_{\substack{x_1 = x_{1o} \\ x_2 = x_{2o}}} (x_1 - x_{1o}) + \left.\frac{\partial f}{\partial x_2}\right|_{\substack{x_1 = x_{1o} \\ x_2 = x_{2o}}} (x_2 - x_{2o}) \right]$$

which can be written as $y = y_o + K_1(x_1 - x_{1o}) + K_2(x_2 - x_{2o})$

where $y_o = f(x_{1o}, x_{2o})$, $K_1 = \left.\frac{\partial f}{\partial x_1}\right|_{\substack{x_1 = x_{1o} \\ x_2 = x_{2o}}}$ and $K_2 = \left.\frac{\partial f}{\partial x_2}\right|_{\substack{x_1 = x_{1o} \\ x_2 = x_{2o}}}$

Finally assuming $\Delta x_1 = x_1 - x_{1o}$, $\Delta x_2 = x_2 - x_{2o}$ and $\Delta y = y - y_o$ yields

$$\Delta y = K_1 \Delta x_1 + K_2 \Delta x_2$$

Linearization of Nonlinear Systems

Linearization of the system $y = f(x_1, \ldots x_n)$ yields

$$\Rightarrow \quad y = f(x_{1o}, \ldots, x_{no}) + \left[\left.\frac{\partial f}{\partial x_1}\right|_{\substack{x_1 = x_{1o}\\ \vdots \\ x_n = x_{no}}} (x_1 - x_{1o}) + \cdots + \left.\frac{\partial f}{\partial x_n}\right|_{\substack{x_1 = x_{1o}\\ \vdots \\ x_n = x_{no}}} (x_n - x_{no}) \right]$$

which can be written as $y = y_o + K_1(x_1 - x_{1o}) + \cdots + K_n(x_n - x_{no})$

where $y_o = f(x_{1o}, \ldots, x_{no})$, $K_1 = \left.\frac{\partial f}{\partial x_1}\right|_{\substack{x_1 = x_{1o}\\ \vdots \\ x_n = x_{no}}}$ and $K_n = \left.\frac{\partial f}{\partial x_n}\right|_{\substack{x_1 = x_{1o}\\ \vdots \\ x_n = x_{no}}}$

Finally assuming $\Delta x_1 = x_1 - x_{1o}$, $\Delta x_n = x_n - x_{no}$ and $\Delta y = y - y_o$ yields

$$\Delta y = K_1 \Delta x_1 + \cdots + K_n \Delta x_n$$

Linearization of Nonlinear Systems

Example: Linearize the following nonlinear algebraic equation about $x_{1o} = 2$ and $x_{2o} = 1$

$$y = 2x_1^2 x_2 + \sin(\pi x_1) + \sqrt{2x_1 x_2}$$

Solution:

$$\Delta y = K_1 \Delta x_1 + K_2 \Delta x_2$$

$$\Delta y = y - y_o$$

$$\Delta x_1 = x_1 - x_{1o}$$

$$\Delta x_2 = x_2 - x_{2o}$$

$$K_1 = \left. \frac{\partial f}{\partial x_1} \right|_{\substack{x_1 = x_{1o} \\ x_2 = x_{2o}}}$$

$$K_2 = \left. \frac{\partial f}{\partial x_2} \right|_{\substack{x_1 = x_{1o} \\ x_2 = x_{2o}}}$$

$$y_o = f(x_{1o}, x_{2o})$$

Linearization of Nonlinear Systems

Solution: $x_{1o} = 2$ $x_{2o} = 1$ $y = 2x_1^2 x_2 + \sin(\pi x_1) + \sqrt{2x_1 x_2}$

$$\Delta y = K_1 \Delta x_1 + K_2 \Delta x_2 \quad \Rightarrow \quad \Delta y = (8.5+\pi)\Delta x_1 + 9\,\Delta x_2$$

$$y_o = f(x_{1o}, x_{2o}) = 2\times 2^2 \times 1 + \sin(2\pi) + \sqrt{2\times 2\times 1} = 10$$

$$\Delta y = y - y_o = y - 10 \qquad \Delta x_1 = x_1 - x_{1o} = x_1 - 2 \qquad \Delta x_2 = x_2 - x_{2o} = x_2 - 1$$

$$K_1 = \left.\frac{\partial f}{\partial x_1}\right|_{\substack{x_1 = x_{1o}\\ x_2 = x_{2o}}} = \left.\left(4x_1x_2 + \pi\cos(\pi x_1) + \frac{x_2}{\sqrt{2x_1x_2}}\right)\right|_{\substack{x_1=2\\ x_2=1}} = 8 + \pi + \frac{1}{2} = (8.5+\pi)$$

$$K_2 = \left.\frac{\partial f}{\partial x_2}\right|_{\substack{x_1 = x_{1o}\\ x_2 = x_{2o}}} = \left.\left(2x_1^2 + \frac{x_1}{\sqrt{2x_1x_2}}\right)\right|_{\substack{x_1=2\\ x_2=1}} = 8 + 1 = 9$$

Linearization of Induction Machine Model

Linearization of electromagnetic torque expression

- Consider the following operating point

$$P_o = \begin{pmatrix} i^e_{qs,o} & i^e_{ds,o} & i'^e_{qr,o} & i'^e_{dr,o} \end{pmatrix}$$

- It is required to linearize the following EM torque expression around the given operating point

$$T_{em} = \frac{3}{2}\frac{P}{2\omega_b} x_m \left(i'_{dr}\, i_{qs} - i'_{qr}\, i_{ds} \right)$$

- The EM torque at the operating point is

$$T_{em}\,|_{P_o} = \frac{3}{2}\frac{P}{2\omega_b} x_m \left(i'_{dr,o}\, i_{qs,o} - i'_{qr,o}\, i_{ds,o} \right)$$

Linearization of Induction Machine Model

Linearization of electromagnetic torque expression

$$T_{em} = \frac{3}{2}\frac{P}{2\omega_b} x_m \left(i'_{dr}\, i_{qs} - i'_{qr}\, i_{ds}\right)$$

$$P_o = \left(i^e_{qs,o} \quad i^e_{ds,o} \quad i'^e_{qr,o} \quad i'^e_{dr,o}\right)$$

$$T_{em}\,|_{P_o} = \frac{3}{2}\frac{P}{2\omega_b} x_m \left(i'_{dr,o}\, i_{qs,o} - i'_{qr,o}\, i_{ds,o}\right)$$

- Using the Taylor series expansion yields

$$T_{em} = T_{em}\,|_{P_o} + \left[\left.\frac{\partial T_{em}}{\partial i^e_{qs}}\right|_{P_o} \Delta i^e_{qs} + \left.\frac{\partial T_{em}}{\partial i^e_{ds}}\right|_{P_o} \Delta i^e_{ds} + \left.\frac{\partial T_{em}}{\partial i'^e_{qr}}\right|_{P_o} \Delta i'^e_{qr} + \left.\frac{\partial T_{em}}{\partial i'^e_{dr}}\right|_{P_o} \Delta i'^e_{dr} \right]$$

$$\Delta T_{em} = \frac{3}{2}\frac{P}{2\omega_b} x_m \left[i'^e_{dr,o}\, \Delta i^e_{qs} - i'^e_{qr,o}\, \Delta i^e_{ds} - i^e_{ds,o}\, \Delta i'^e_{qr} + i^e_{qs,o}\, \Delta i'^e_{dr} \right]$$

Linearization of Induction Machine Model

Linearization of electromagnetic torque expression

$$\Delta T_{em} = \frac{3}{2}\frac{P}{2\omega_b} x_m \left[i'^{e}_{dr,o}\,\Delta i^{e}_{qs} - i'^{e}_{qr,o}\,\Delta i^{e}_{ds} - i^{e}_{ds,o}\,\Delta i'^{e}_{qr} + i^{e}_{qs,o}\,\Delta i'^{e}_{dr} \right]$$

where

$$\Delta T_{em} = T_{em} - T_{em}\,|_{P_o}$$

$$T_{em}\,|_{P_o} = \frac{3}{2}\frac{P}{2\omega_b} x_m \left(i'_{dr,o}\, i_{qs,o} - i'_{qr,o}\, i_{ds,o} \right)$$

$$\Delta i^{e}_{qs} = i^{e}_{qs} - i^{e}_{qs,o}$$

$$\Delta i^{e}_{ds} = i^{e}_{ds} - i^{e}_{ds,o}$$

$$\Delta i'^{e}_{qr} = i'^{e}_{qr} - i'^{e}_{qr,o}$$

$$\Delta i'^{e}_{dr} = i'^{e}_{dr} - i'^{e}_{dr,o}$$

Second Technique:

Perturbation Method

Linearization of Nonlinear Systems

- Consider a **nonlinear model** in the following form

$$\begin{cases} \mathbf{f}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = 0 \\ \mathbf{y} = \mathbf{g}(\mathbf{x}, \mathbf{u}) \end{cases}$$

where $\mathbf{x}$ is the vector of state variables, $\mathbf{u}$ is the input vector and $\mathbf{y}$ is the output vector.

- Assume the system is **operating in an equilibrium point** in which $\mathbf{x}=\mathbf{x}_o$, $\mathbf{u}=\mathbf{u}_o$ and $\mathbf{y}=\mathbf{y}_o$.
- When a **small displacement**, denoted by Δ, is applied to each component of $\mathbf{x}$, $\mathbf{u}$ and $\mathbf{y}$ vectors, the perturbed variables will still satisfy the governing differential-algebraic equations:

$$\begin{cases} \mathbf{f}\left(\dot{\mathbf{x}}_{\mathbf{x}=\mathbf{x}_o} + \Delta\dot{\mathbf{x}},\ \ \mathbf{x}_o + \Delta\mathbf{x},\ \ \mathbf{u}_o + \Delta\mathbf{u}\right) = 0 \\ \mathbf{y}_o + \Delta\mathbf{y} = \mathbf{g}(\mathbf{x}_o + \Delta\mathbf{x},\ \ \mathbf{u}_o + \Delta\mathbf{u}) \end{cases}$$

Linearization of Nonlinear Systems

$$\begin{cases} \mathbf{f}\left(\dot{\mathbf{x}}_{\mathbf{x}=\mathbf{x}_o} + \Delta\dot{\mathbf{x}},\ \ \mathbf{x}_o + \Delta\mathbf{x},\ \ \mathbf{u}_o + \Delta\mathbf{u}\right) = 0 \\ \mathbf{y}_o + \Delta\mathbf{y} = \mathbf{g}\left(\mathbf{x}_o + \Delta\mathbf{x},\ \ \mathbf{u}_o + \Delta\mathbf{u}\right) \end{cases}$$

- At the equilibrium point we have

$$\dot{\mathbf{x}}_{\mathbf{x}=\mathbf{x}_o} = \mathbf{0}$$

$$\mathbf{f}\left(0, \mathbf{x}_o, \mathbf{u}_o\right) = 0$$

- Substituting these **two relations** into the **perturbed state-space** equations and **neglecting higher order Δ terms** can result in the following **linear** state-space equations:

$$\begin{cases} \Delta\dot{\mathbf{x}} = \mathbf{A}\Delta\mathbf{x} + \mathbf{B}\Delta\mathbf{u} \\ \Delta\mathbf{y} = \mathbf{C}\Delta\mathbf{x} + \mathbf{D}\Delta\mathbf{u} \end{cases}$$

Linearization of Induction Machine Model

Linearization of electromagnetic torque expression

- Consider the following operating point

$$P_o = \left(i_{qs,o}^{e} \quad i_{ds,o}^{e} \quad i_{qr,o}^{\prime e} \quad i_{dr,o}^{\prime e} \right)$$

- It is required to linearize the following EM torque expression around the given operating point

$$T_{em} = \frac{3}{2}\frac{P}{2\omega_b} x_m \left(i'_{dr}\, i_{qs} - i'_{qr}\, i_{ds} \right)$$

- The EM torque at the operating point is

$$T_{em}\,|_{P_o} = \frac{3}{2}\frac{P}{2\omega_b} x_m \left(i'_{dr,o}\, i_{qs,o} - i'_{qr,o}\, i_{ds,o} \right)$$

Linearization of Induction Machine Model

Linearization of electromagnetic torque expression

$$T_{em} = \frac{3}{2}\frac{P}{2\omega_b} x_m \left(i'_{dr}\, i_{qs} - i'_{qr}\, i_{ds}\right)$$

$$P_o = \left(i^e_{qs,o} \quad i^e_{ds,o} \quad i'^e_{qr,o} \quad i'^e_{dr,o}\right)$$

$$T_{em}\,|_{P_o} = \frac{3}{2}\frac{P}{2\omega_b} x_m \left(i'_{dr,o}\, i_{qs,o} - i'_{qr,o}\, i_{ds,o}\right)$$

- Using the perturbation method yields

$$T_{em}\,|_{P_o} + \Delta T_{em} = \frac{3}{2}\frac{P}{2\omega_b} x_m \left[\left(i'_{dr,o} + \Delta i'_{dr}\right)\left(i_{qs,o} + \Delta i_{qs}\right) - \left(i'_{qr,o} + \Delta i'_{qr}\right)\left(i_{ds,o} + \Delta i_{ds}\right)\right]$$

$$\Delta T_{em} = \frac{3}{2}\frac{P}{2\omega_b} x_m \left[i'^e_{dr,o}\, \Delta i^e_{qs} - i'^e_{qr,o}\, \Delta i^e_{ds} - i^e_{ds,o}\, \Delta i'^e_{qr} + i^e_{qs,o}\, \Delta i'^e_{dr}\right]$$

Linearization of Induction Machine Model

- By using the **perturbation** method, **linearize** the following nonlinear IM model which is in the synchronous *qd*0 RF

$$\begin{bmatrix} v_{qs}^e \\ v_{ds}^e \\ v_{qr}'^e \\ v_{dr}'^e \end{bmatrix} = \begin{bmatrix} r_s + \frac{p}{\omega_b} x_{ss} & \frac{\omega_e}{\omega_b} x_{ss} & \frac{p}{\omega_b} x_m & \frac{\omega_e}{\omega_b} x_m \\ -\frac{\omega_e}{\omega_b} x_{ss} & r_s + \frac{p}{\omega_b} x_{ss} & -\frac{\omega_e}{\omega_b} x_m & \frac{p}{\omega_b} x_m \\ \frac{p}{\omega_b} x_m & s\frac{\omega_e}{\omega_b} x_m & r_r' + \frac{p}{\omega_b} x_{rr}' & s\frac{\omega_e}{\omega_b} x_{rr}' \\ -s\frac{\omega_e}{\omega_b} x_m & \frac{p}{\omega_b} x_m & -s\frac{\omega_e}{\omega_b} x_{rr}' & r_r' + \frac{p}{\omega_b} x_{rr}' \end{bmatrix} \begin{bmatrix} i_{qs}^e \\ i_{ds}^e \\ i_{qr}'^e \\ i_{dr}'^e \end{bmatrix}$$

Nonlinear terms are encircled.

$$T_{em} - T_{mech} = \frac{2J}{P} \frac{d\omega_r}{dt}$$

$$T_{em} = \frac{3}{2} \frac{P}{2\omega_b} x_m \left(i_{dr}' \, i_{qs} - i_{qr}' \, i_{ds} \right)$$

$$s = \frac{\omega_e - \omega_r}{\omega_e}$$

Linearization of Induction Machine Model

- Clearly the input and state vectors are as follows:

$$\mathbf{x} = \begin{bmatrix} i_{qs}^{e} & i_{ds}^{e} & i_{qr}'^{e} & i_{dr}'^{e} & \omega_r \end{bmatrix}^T$$

$$\mathbf{u} = \begin{bmatrix} v_{qs}^{e} & v_{ds}^{e} & v_{qr}'^{e} & v_{dr}'^{e} & T_{mech} \end{bmatrix}^T$$

- The linearization is performed around an equilibrium point:

$$\mathbf{x}_o = \begin{bmatrix} i_{qs,o}^{e} & i_{ds,o}^{e} & i_{qr,o}'^{e} & i_{dr,o}'^{e} & \omega_{r,o} \end{bmatrix}^T$$

$$\mathbf{u}_o = \begin{bmatrix} v_{qs,o}^{e} & v_{ds,o}^{e} & v_{qr,o}'^{e} & v_{dr,o}'^{e} & T_{mech,o} \end{bmatrix}^T$$

- The small displacement is introduced for each variables:

$$\Delta\mathbf{x} = \begin{bmatrix} \Delta i_{qs}^{e} & \Delta i_{ds}^{e} & \Delta i_{qr}'^{e} & \Delta i_{dr}'^{e} & \Delta\omega_r \end{bmatrix}^T$$

$$\Delta\mathbf{u} = \begin{bmatrix} \Delta v_{qs}^{e} & \Delta v_{ds}^{e} & \Delta v_{qr}'^{e} & \Delta v_{dr}'^{e} & \Delta T_{mech} \end{bmatrix}^T$$

where e.g. $\Delta i_{qs}^{e} = i_{qs}^{e} - i_{qs,o}^{e}$ $\quad \Delta i_{ds}^{e} = i_{ds}^{e} - i_{ds,o}^{e}$

Linearization of Induction Machine Model

- Perturbing the governing equations and neglecting the higher order Δ terms yields

$$\begin{bmatrix} \Delta v_{qs}^{e} \\ \Delta v_{ds}^{e} \\ \Delta v_{qr}^{\prime e} \\ \Delta v_{dr}^{\prime e} \\ \Delta T_{mech} \end{bmatrix} = \begin{bmatrix} r_s + \frac{p}{\omega_b} x_{ss} & \frac{\omega_e}{\omega_b} x_{ss} & \frac{p}{\omega_b} x_m & \frac{\omega_e}{\omega_b} x_m & 0 \\ -\frac{\omega_e}{\omega_b} x_{ss} & r_s + \frac{p}{\omega_b} x_{ss} & -\frac{\omega_e}{\omega_b} x_m & \frac{p}{\omega_b} x_m & 0 \\ \frac{p}{\omega_b} x_m & s_o \frac{\omega_e}{\omega_b} x_m & r_r' + \frac{p}{\omega_b} x_{rr}' & s_o \frac{\omega_e}{\omega_b} x_{rr}' & -\frac{x_m}{\omega_b} i_{ds,o} - \frac{x_{rr}'}{\omega_b} i_{dr,o}' \\ -s_o \frac{\omega_e}{\omega_b} x_m & \frac{p}{\omega_b} x_m & -s_o \frac{\omega_e}{\omega_b} x_{rr}' & r_r' + \frac{p}{\omega_b} x_{rr}' & \frac{x_m}{\omega_b} i_{qs,o} + \frac{x_{rr}'}{\omega_b} i_{qr,o}' \\ \frac{3P}{4\omega_b} x_m i_{dr,o}' & -\frac{3P}{4\omega_b} x_m i_{qr,o}' & -\frac{3P}{4\omega_b} x_m i_{ds,o} & \frac{3P}{4\omega_b} x_m i_{qs,o} & -\frac{2J}{P} p \end{bmatrix} \begin{bmatrix} \Delta i_{qs}^{e} \\ \Delta i_{ds}^{e} \\ \Delta i_{qr}^{\prime e} \\ \Delta i_{dr}^{\prime e} \\ \Delta \omega_r \end{bmatrix}$$

$$s_o = \frac{\omega_e - \omega_{r,o}}{\omega_e}$$

$$\Delta s = s - s_o = \frac{-\Delta \omega_r}{\omega_e}$$

Linearization of Induction Machine Model

- The obtained governing equations are rewritten in the following form

$$\Delta \mathbf{u} = \mathbf{E}\Delta\dot{\mathbf{x}} - \mathbf{F}\Delta\mathbf{x}$$

where

$$\Delta \mathbf{x} = \begin{bmatrix} \Delta i_{qs}^{e} & \Delta i_{ds}^{e} & \Delta i_{qr}'^{e} & \Delta i_{dr}'^{e} & \Delta \omega_r \end{bmatrix}^T$$

$$\Delta \mathbf{u} = \begin{bmatrix} \Delta v_{qs}^{e} & \Delta v_{ds}^{e} & \Delta v_{qr}'^{e} & \Delta v_{dr}'^{e} & \Delta T_{mech} \end{bmatrix}^T$$

Linearization of Induction Machine Model

- and $\mathbf{E}$ and $\mathbf{F}$ are two constant matrices

$$\mathbf{E}=\frac{1}{\omega_b}\begin{bmatrix} x_{ss} & 0 & x_m & 0 & 0 \\ 0 & x_{ss} & 0 & x_m & 0 \\ x_m & 0 & x'_{rr} & 0 & 0 \\ 0 & x_m & 0 & x'_{rr} & 0 \\ 0 & 0 & 0 & 0 & -\frac{2J\omega_b}{P} \end{bmatrix}$$

$$\boldsymbol{F}=-\begin{bmatrix} r_s & \frac{\omega_e}{\omega_b}x_{ss} & 0 & \frac{\omega_e}{\omega_b}x_m & 0 \\ -\frac{\omega_e}{\omega_b}x_{ss} & r_s & -\frac{\omega_e}{\omega_b}x_m & 0 & 0 \\ 0 & s_o\frac{\omega_e}{\omega_b}x_m & r'_r & s_o\frac{\omega_e}{\omega_b}x'_{rr} & -\frac{x_m}{\omega_b}i_{ds,o}-\frac{x'_{rr}}{\omega_b}i'_{dr,o} \\ -s_o\frac{\omega_e}{\omega_b}x_m & 0 & -s_o\frac{\omega_e}{\omega_b}x'_{rr} & r'_r & \frac{x_m}{\omega_b}i_{qs,o}+\frac{x'_{rr}}{\omega_b}i'_{qr,o} \\ \frac{3P}{4\omega_b}x_m i'_{dr,o} & -\frac{3P}{4\omega_b}x_m i'_{qr,o} & -\frac{3P}{4\omega_b}x_m i_{ds,o} & \frac{3P}{4\omega_b}x_m i_{qs,o} & 0 \end{bmatrix}$$

Linearization of Induction Machine Model

- It is now possible to write the linear state-space equations in the following standard form

$$\Delta\dot{\mathbf{x}} = \mathbf{A}\Delta\mathbf{x} + \mathbf{B}\Delta\mathbf{u}$$

where $\mathbf{A} = \mathbf{E}^{-1}\mathbf{F}$ $\mathbf{B} = \mathbf{E}^{-1}$