In The Name of God The Most Compassionate, The Most Merciful

General Theory of Electric Machines

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Chapter 3
Reference-Frame Theory

Introduction

Some of the applications of the **Mathematical Transformations** are as follows:

1. To **decouple variables**;

2. To **facilitate the solution** of differential equations with time-varying coefficients;

3. To **refer all variables** to a common reference frame.

Fortescue's Transformation

- This transformation is known as the method of **symmetrical components** and developed by Fortescue.
- This transformation states that N **unbalanced phasors** can be represented by N **systems of** N **balanced phasors**.
- It uses a **complex transformation** to decouple the abc phase variables.
- The method of symmetrical components is used to **simplify** analysis of **unbalanced three phase** power systems under both normal and abnormal conditions.
- It is used to decouple an unbalanced three-phase network into **three simpler sequence** (zero, positive and negative) networks.

Fortescue's Transformation

- The method of **symmetrical components** is expressed as follows

$$[\mathbf{f}_{012}] = [\mathbf{T}_{012}][\mathbf{f}_{abc}]$$

$$[\mathbf{f}_{abc}] = [\mathbf{T}_{012}]^{-1}[\mathbf{f}_{012}]$$

$$[\mathbf{f}_{012}] = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \end{bmatrix}$$

$$[\mathbf{f}_{abc}] = \begin{bmatrix} f_a \\ f_b \\ f_c \end{bmatrix}$$

- Variable f may be the **currents**, **voltages** or **fluxes** and the transformation and its inverse are given by

$$[\mathbf{T}_{012}] = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix}$$

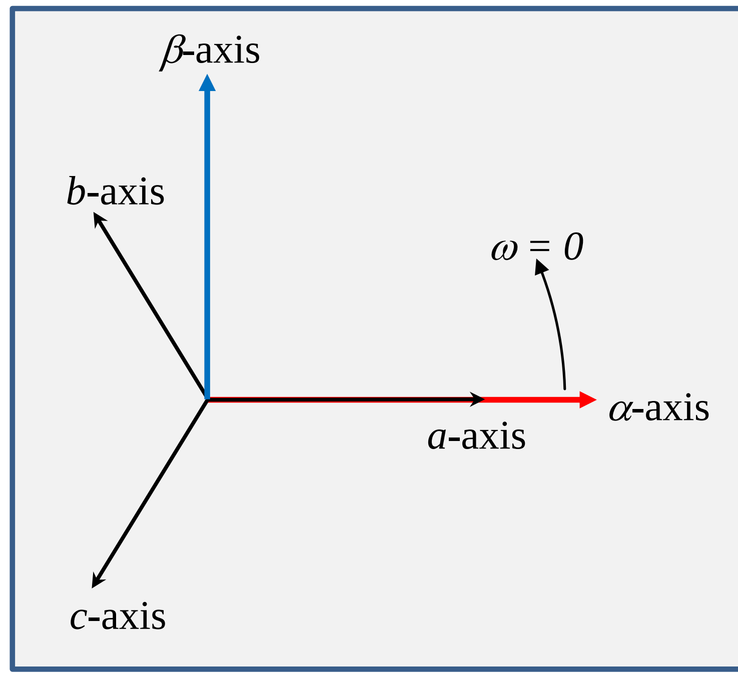
$$[\mathbf{T}_{012}]^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix}$$

where

$$a = e^{j\frac{2\pi}{3}}$$

Clarke's Transformation

- The **stationary** two-phase variables of Clarke's transformation are denoted as α and β.
- As shown below, the α-axis coincides with the phase a-axis and the β-axis leads the α-axis by $\pi/2$.
- A third variable known as the **zero-sequence component** is also included.
- Clarke's transformation is **not power-invariant** (i.e. the values of power before and after the transformation are not the same.



Clarke's Transformation

- Clarke's transformation is expressed as follows

$$[\mathbf{f}_{\alpha\beta 0}] = [\mathbf{T}_{\alpha\beta 0}]\,[\mathbf{f}_{abc}]$$

$$[\mathbf{f}_{abc}] = [\mathbf{T}_{\alpha\beta 0}]^{-1}[\mathbf{f}_{\alpha\beta 0}]$$

$$[\mathbf{f}_{\alpha\beta 0}] = \begin{bmatrix} f_\alpha \\ f_\beta \\ f_0 \end{bmatrix}$$

$$[\mathbf{f}_{abc}] = \begin{bmatrix} f_a \\ f_b \\ f_c \end{bmatrix}$$

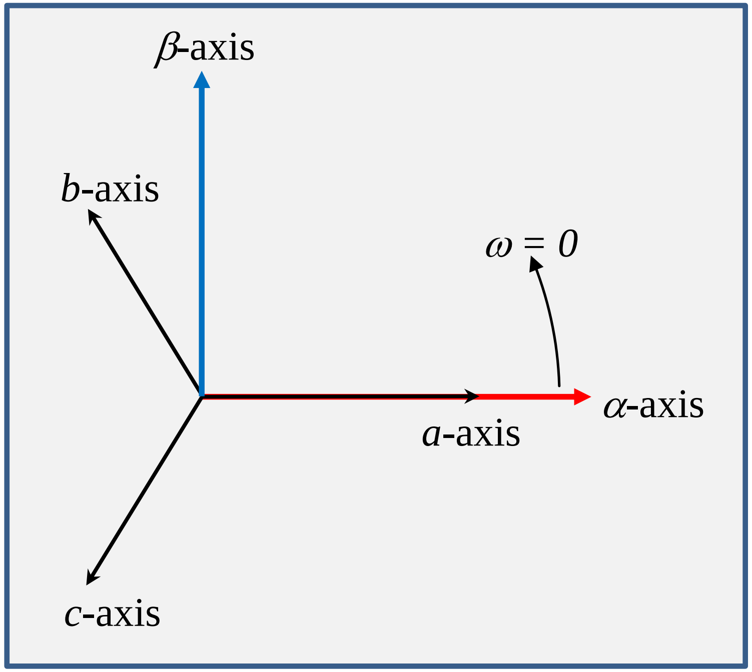
- Similarly variable *f* may be the **currents**, **voltages** or **fluxes** and the transformation and its inverse are given by

$$[\mathbf{T}_{\alpha\beta 0}] = \frac{2}{3}\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$[\mathbf{T}_{\alpha\beta 0}]^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 1 \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 1 \end{bmatrix}$$

Concordia's Transformation

- Concordia's transformation is **similar** to Clarke's transformation.
- The only difference is that Concordia's transformation is **power-invariant** (i.e. the values of power before and after the transformation are identical.
- To have the power-invariant property, the transformation matrix must be **orthogonal**.
- A matrix is orthogonal if its **inverse** and its **transpose** are the same, i.e.
- $\mathbf{M}$ is orthogonal if $\mathbf{M}^{-1} = \mathbf{M}^{T}$



Concordia's Transformation

- Concordia's transformation is similarly expressed as follows

$$[\mathbf{f}_{\alpha\beta 0}] = [\mathbf{T}_{\alpha\beta 0}]\,[\mathbf{f}_{abc}]$$

$$[\mathbf{f}_{abc}] = [\mathbf{T}_{\alpha\beta 0}]^{-1}\,[\mathbf{f}_{\alpha\beta 0}]$$

$$[\mathbf{f}_{\alpha\beta 0}] = \begin{bmatrix} f_\alpha \\ f_\beta \\ f_0 \end{bmatrix}$$

$$[\mathbf{f}_{abc}] = \begin{bmatrix} f_a \\ f_b \\ f_c \end{bmatrix}$$

- The transformation and its inverse are given by

$$[\mathbf{T}_{\alpha\beta 0}] = \sqrt{\frac{2}{3}} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$[\mathbf{T}_{\alpha\beta 0}]^{-1} = \sqrt{\frac{2}{3}} \begin{bmatrix} 1 & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Power-Invariant Property

Example: Consider a balanced 3-phase system with ohmic load. Show that:

1. Clarke's transformation is not power-invariant,

2. Concordia's transformation is power-invariant.

$$\begin{cases} v_a = V_m \cos(\omega t) \\ v_b = V_m \cos(\omega t - 2\pi/3) \\ v_c = V_m \cos(\omega t - 4\pi/3) \end{cases}$$

$$\begin{cases} i_a = I_m \cos(\omega t) \\ i_b = I_m \cos(\omega t - 2\pi/3) \\ i_c = I_m \cos(\omega t - 4\pi/3) \end{cases}$$

Power-Invariant Property

Example: Part 1) Clarke's transformation is not power-invariant,

- Using the 3-phase expressions at $\omega t = 0$

$$\begin{cases} v_a = V_m \\ v_b = \frac{-1}{2} V_m \\ v_c = \frac{-1}{2} V_m \end{cases} \qquad \begin{cases} i_a = I_m \\ i_b = \frac{-1}{2} I_m \\ i_c = \frac{-1}{2} I_m \end{cases} \Rightarrow \quad P = v_a\, i_a + v_b\, i_b + v_c\, i_c = \tfrac{3}{2} V_m I_m$$

Therefore not power-invariant

- Using Clarke's transformation at $\omega t = 0$

$$\begin{cases} v_\alpha = V_m \\ v_\beta = 0 \\ v_0 = 0 \end{cases} \qquad \begin{cases} i_\alpha = I_m \\ i_\beta = 0 \\ i_0 = 0 \end{cases} \Rightarrow \quad P = v_\alpha\, i_\alpha + v_\beta\, i_\beta + v_0\, i_0 = V_m I_m$$

Power-Invariant Property

Example: Part 2) Concordia's transformation is power-invariant,

- Using the 3-phase expressions at $\omega t = 0$

$$\begin{cases} v_a = V_m \\ v_b = \frac{-1}{2} V_m \\ v_c = \frac{-1}{2} V_m \end{cases} \quad \begin{cases} i_a = I_m \\ i_b = \frac{-1}{2} I_m \\ i_c = \frac{-1}{2} I_m \end{cases} \Rightarrow P = v_a i_a + v_b i_b + v_c i_c = \frac{3}{2} V_m I_m$$

- Using Concordia's transformation at $\omega t = 0$

$$\begin{cases} v_\alpha = \sqrt{\frac{3}{2}} V_m \\ v_\beta = 0 \\ v_0 = 0 \end{cases} \quad \begin{cases} i_\alpha = \sqrt{\frac{3}{2}} I_m \\ i_\beta = 0 \\ i_0 = 0 \end{cases} \Rightarrow P = v_\alpha i_\alpha + v_\beta i_\beta + v_0 i_0 = \frac{3}{2} V_m I_m$$

Therefore power-invariant

n-phase to 2-phase Transformation

- Another commonly-used transformation is the **polyphase to orthogonal two-phase transformation**.

- For the n-phase to two-phase case, it is expressed as

$$[\mathbf{f}_{xy}] = [\mathbf{T}(\theta)][\mathbf{f}_{123\ldots n}]$$

where

$$[\boldsymbol{T}(\theta)] = \sqrt{\frac{2}{n}}\begin{bmatrix} \cos\theta & \cos(\theta-\alpha) & \cdots & \cos(\theta-(n-1)\alpha) \\ \sin\theta & \sin(\theta-\alpha) & \cdots & \sin(\theta-(n-1)\alpha) \end{bmatrix}$$

and α is the *electrical* angle between adjacent magnetic axes of the uniformly distributed n-phase winding. The coefficient $\sqrt{2/n}$ is to make the transformation **power-invariant**.

Park's Transformation

- Park's transformation is a well-known 3-phase to 2-phase transformation in **synchronous machine** analysis.

- Three different cases are introduced:

Case 1: The q-axis is **leading** the d-axis by 90 electrical degrees; and the angle between the **d-axis** w.r.t. the a-axis is used.

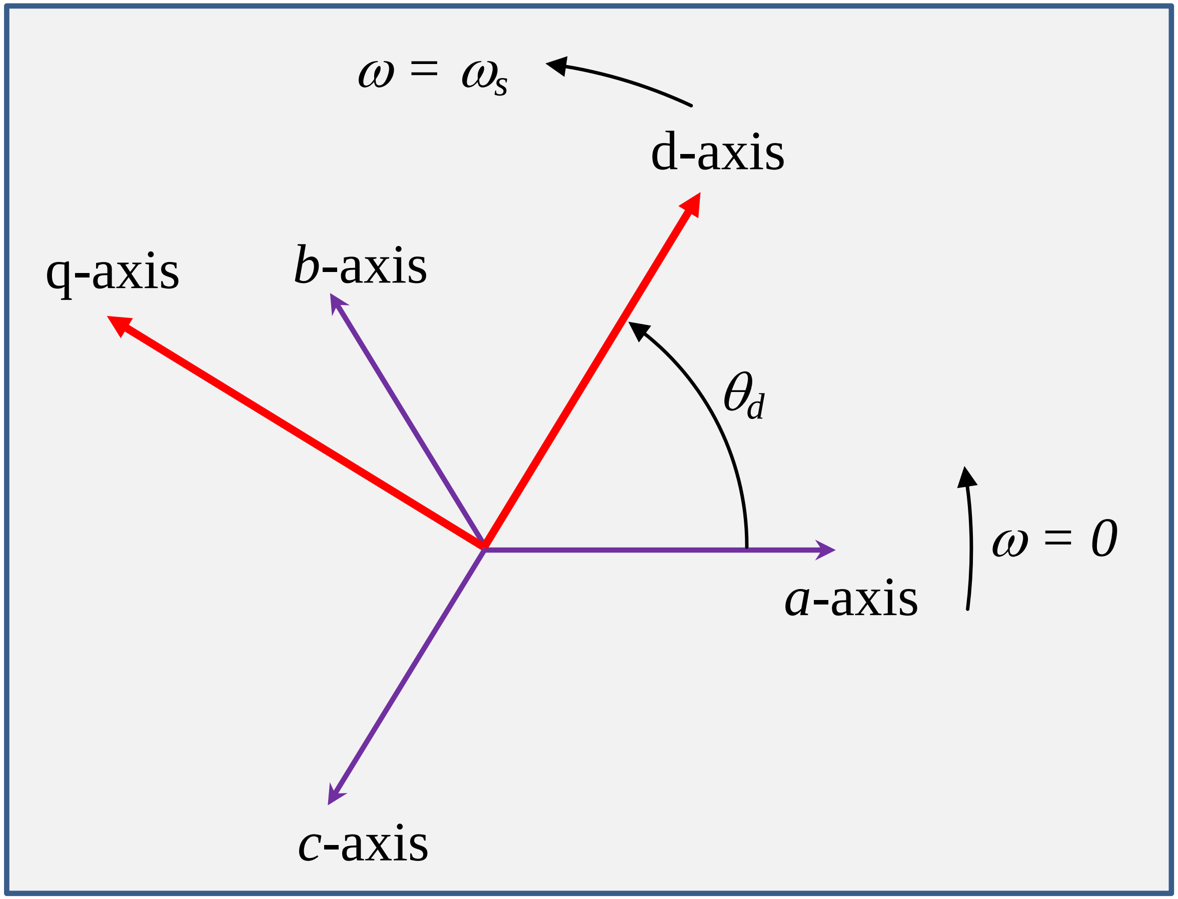
Case 2: The q-axis is **lagging** the d-axis by 90 electrical degrees; and the angle between the **d-axis** w.r.t. the a-axis is used.

Case 3: The q-axis is **leading** the d-axis by 90 electrical degrees; and the angle between the **q-axis** w.r.t. the a-axis is used.

Park's Transformation

Case 1

Case 1: The q-axis is leading the d-axis by 90 electrical degrees; and the angle between the d-axis w.r.t. the *a*-axis is used.



Generator Notation

Park's Transformation

Case 1

- The **case 1** of **Park's transformation** is expressed as:

$$\left[\mathbf{f}_{dq0}\right] = \left[\mathbf{T}_{dq0}(\theta_d)\right]\left[\mathbf{f}_{abc}\right]$$

$$\left[\mathbf{f}_{dq0}\right] = \begin{bmatrix} f_d \\ f_q \\ f_0 \end{bmatrix}$$

$$\left[\mathbf{f}_{abc}\right] = \begin{bmatrix} f_a \\ f_b \\ f_c \end{bmatrix}$$

where

$$\left[\mathbf{T}_{dq0}(\theta_d)\right] = \frac{2}{3}\begin{bmatrix} \cos\theta_d & \cos\left(\theta_d - 2\pi/3\right) & \cos\left(\theta_d + 2\pi/3\right) \\ -\sin\theta_d & -\sin\left(\theta_d - 2\pi/3\right) & -\sin\left(\theta_d + 2\pi/3\right) \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\theta_d = \omega t + \theta_0$$

Park's Transformation

Case 1

- The **case 1** of **inverse Park's transformation** is expressed as:

$$[\mathbf{f}_{abc}] = [\mathbf{T}_{dq0}(\theta_d)]^{-1} [\mathbf{f}_{dqo}]$$

$$[\mathbf{f}_{dq0}] = \begin{bmatrix} f_d \\ f_q \\ f_0 \end{bmatrix}$$

$$[\mathbf{f}_{abc}] = \begin{bmatrix} f_a \\ f_b \\ f_c \end{bmatrix}$$

where

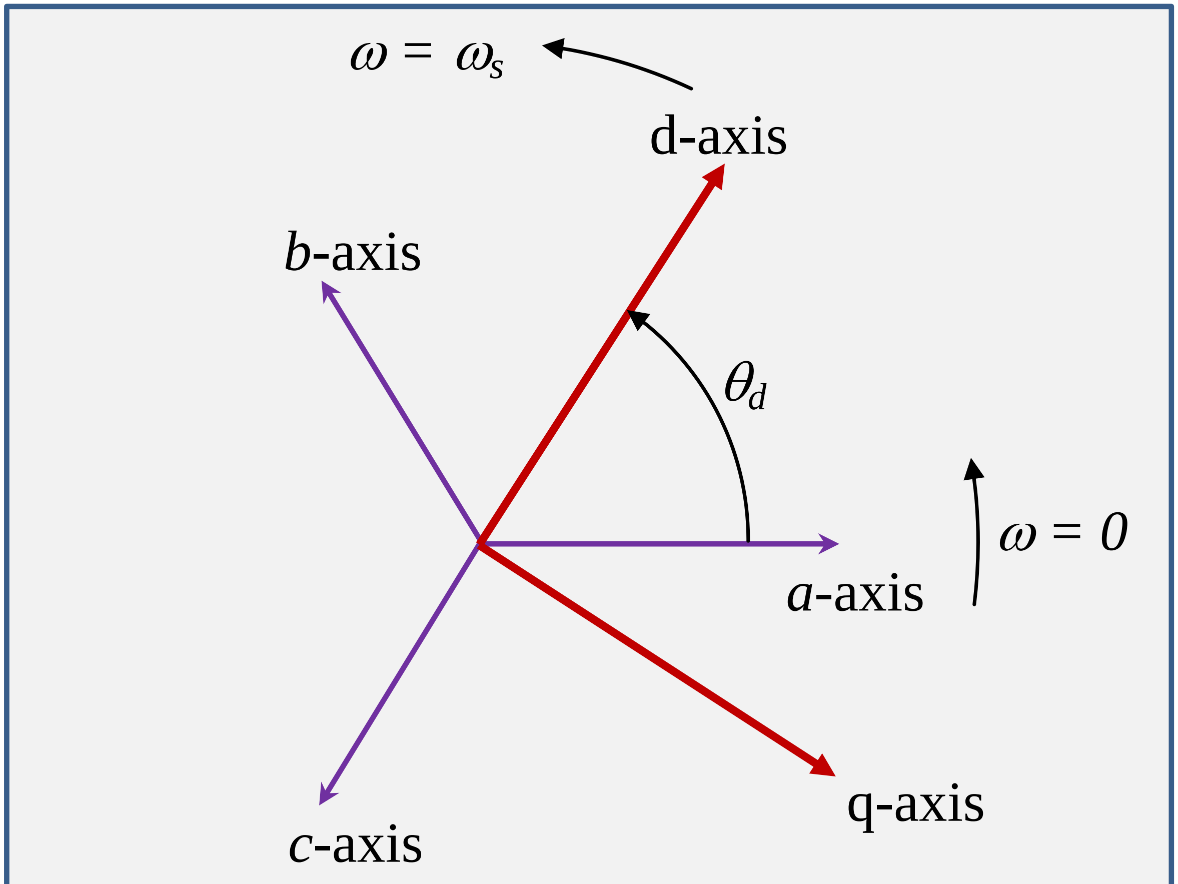
$$[\mathbf{T}_{dq0}(\theta_d)]^{-1} = \begin{bmatrix} \cos\theta_d & -\sin\theta_d & 1 \\ \cos(\theta_d - 2\pi/3) & -\sin(\theta_d - 2\pi/3) & 1 \\ \cos(\theta_d + 2\pi/3) & -\sin(\theta_d + 2\pi/3) & 1 \end{bmatrix}$$

$$\theta_d = \omega t + \theta_0$$

Park's Transformation

Case 2

Case 2: The q-axis is **lagging** the d-axis by 90 electrical degrees; and the angle between the **d-axis** w.r.t. the *a*-axis is used.



Motor Notation

Park's Transformation

Case 2

- The **case 2** of **Park's transformation** is expressed as:

$$\left[\mathbf{f}_{dq0}\right] = \left[\mathbf{T}_{dq0}(\theta_d)\right]\left[\mathbf{f}_{abc}\right]$$

$$\left[\mathbf{f}_{dq0}\right] = \begin{bmatrix} f_d \\ f_q \\ f_0 \end{bmatrix}$$

$$\left[\mathbf{f}_{abc}\right] = \begin{bmatrix} f_a \\ f_b \\ f_c \end{bmatrix}$$

where

$$\left[\mathbf{T}_{dq0}(\theta_d)\right] = \frac{2}{3}\begin{bmatrix} \cos\theta_d & \cos\left(\theta_d - 2\pi/3\right) & \cos\left(\theta_d + 2\pi/3\right) \\ \sin\theta_d & \sin\left(\theta_d - 2\pi/3\right) & \sin\left(\theta_d + 2\pi/3\right) \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\theta_d = \omega t + \theta_0$$

Park's Transformation

Case 2

- The **case 2** of **inverse Park's transformation** is expressed as:

$$[\mathbf{f}_{abc}] = [\mathbf{T}_{dq0}(\theta_d)]^{-1}[\mathbf{f}_{dqo}]$$

$$[\mathbf{f}_{dq0}] = \begin{bmatrix} f_d \\ f_q \\ f_0 \end{bmatrix}$$

$$[\mathbf{f}_{abc}] = \begin{bmatrix} f_a \\ f_b \\ f_c \end{bmatrix}$$

where

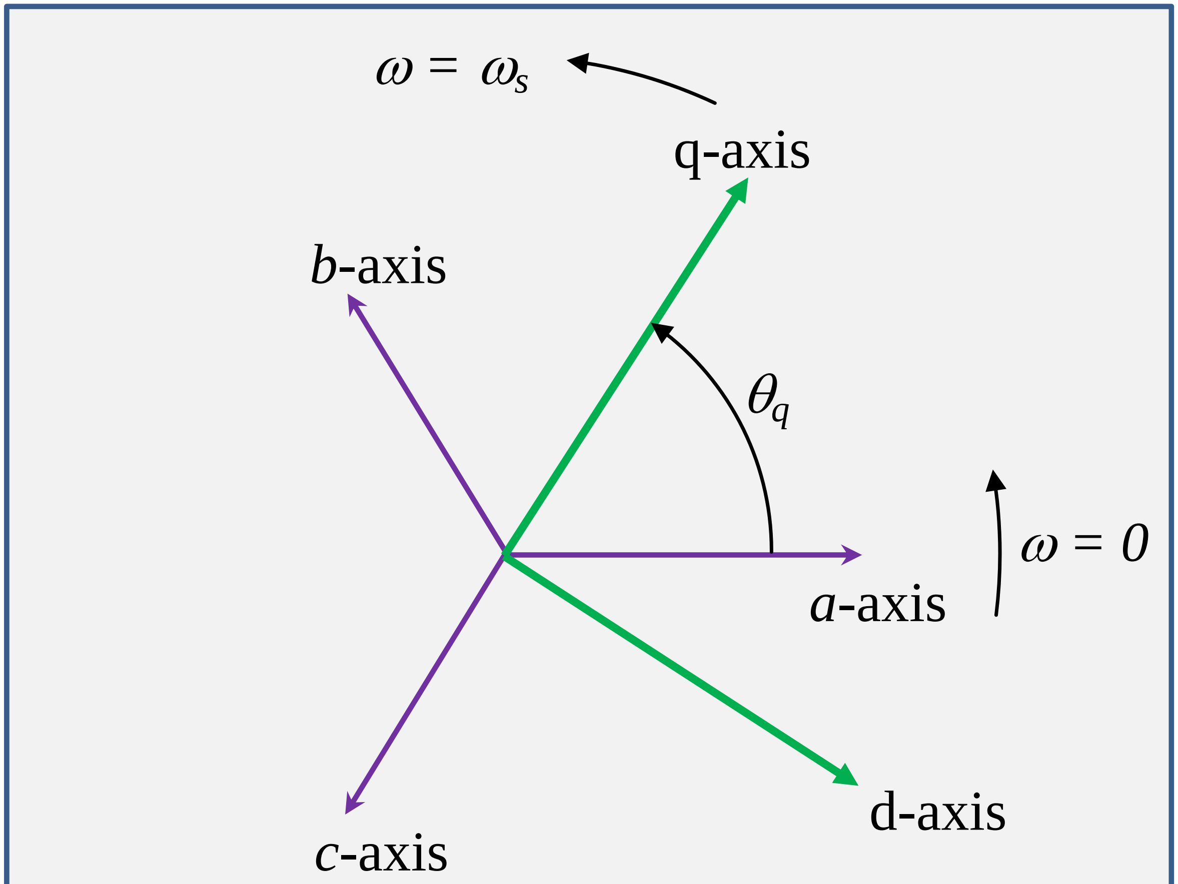
$$[\mathbf{T}_{dq0}(\theta_d)]^{-1} = \begin{bmatrix} \cos\theta_d & \sin\theta_d & 1 \\ \cos(\theta_d - 2\pi/3) & \sin(\theta_d - 2\pi/3) & 1 \\ \cos(\theta_d + 2\pi/3) & \sin(\theta_d + 2\pi/3) & 1 \end{bmatrix}$$

$$\theta_d = \omega t + \theta_0$$

Park's Transformation

Case 3

Case 3: The q-axis is **leading** the d-axis by 90 electrical degrees; and the angle between the **q-axis** w.r.t. the *a*-axis is used.



Motor Notation

Park's Transformation

Case 3

- The **case 3** of **Park's transformation** is expressed as:

$$[\mathbf{f}_{qd0}] = [\mathbf{T}_{qd0}(\theta_q)]\,[\mathbf{f}_{abc}]$$

$$[\mathbf{f}_{qd0}] = \begin{bmatrix} f_q \\ f_d \\ f_0 \end{bmatrix}$$

$$[\mathbf{f}_{abc}] = \begin{bmatrix} f_a \\ f_b \\ f_c \end{bmatrix}$$

where

$$[\mathbf{T}_{qd0}(\theta_q)] = \frac{2}{3}\begin{bmatrix} \cos\theta_q & \cos\left(\theta_q - 2\pi/3\right) & \cos\left(\theta_q + 2\pi/3\right) \\ \sin\theta_q & \sin\left(\theta_q - 2\pi/3\right) & \sin\left(\theta_q + 2\pi/3\right) \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\theta_q = \omega t + \theta_0'$$

$$\theta_q = \theta_d + \tfrac{\pi}{2}$$

Park's Transformation

Case 3

- The **case 3** of **inverse Park's transformation** is expressed as:

$$\left[\mathbf{f}_{abc}\right] = \left[\mathbf{T}_{qd0}(\theta_q)\right]^{-1}\left[\mathbf{f}_{qd0}\right]$$

$$\left[\mathbf{f}_{qd0}\right] = \begin{bmatrix} f_q \\ f_d \\ f_0 \end{bmatrix}$$

$$\left[\mathbf{f}_{abc}\right] = \begin{bmatrix} f_a \\ f_b \\ f_c \end{bmatrix}$$

where

$$\left[\mathbf{T}_{qd0}(\theta_q)\right]^{-1} = \begin{bmatrix} \cos\theta_q & \sin\theta_q & 1 \\ \cos\left(\theta_q - 2\pi/3\right) & \sin\left(\theta_q - 2\pi/3\right) & 1 \\ \cos\left(\theta_q + 2\pi/3\right) & \sin\left(\theta_q + 2\pi/3\right) & 1 \end{bmatrix}$$

$$\theta_q = \omega t + \theta_0'$$

$$\theta_q = \theta_d + \tfrac{\pi}{2}$$

Park's Transformation on a 3-phase Sinusoidal System

- Consider the following 3-phase voltage:

$$[\mathbf{v}_{abc}] = \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} = \begin{bmatrix} V_m \cos(\omega t) \\ V_m \cos(\omega t - 2\pi/3) \\ V_m \cos(\omega t - 4\pi/3) \end{bmatrix}$$

- The aim is to find the case 3 of Park's transformation.

$$[\mathbf{v}_{qd0}] = [\mathbf{T}_{qd0}(\theta_q)]\ [\mathbf{v}_{abc}]$$

$$[\mathbf{T}_{qd0}(\theta_q)] = \frac{2}{3}\begin{bmatrix} \cos\theta_q & \cos\left(\theta_q - 2\pi/3\right) & \cos\left(\theta_q + 2\pi/3\right) \\ \sin\theta_q & \sin\left(\theta_q - 2\pi/3\right) & \sin\left(\theta_q + 2\pi/3\right) \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Park's Transformation on a 3-phase Sinusoidal System

- Therefore

$$[\mathbf{v}_{qd0}] = \frac{2}{3}\begin{bmatrix} \cos\theta_q & \cos(\theta_q - 2\pi/3) & \cos(\theta_q + 2\pi/3) \\ \sin\theta_q & \sin(\theta_q - 2\pi/3) & \sin(\theta_q + 2\pi/3) \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} V_m\cos(\omega t) \\ V_m\cos(\omega t - 2\pi/3) \\ V_m\cos(\omega t - 4\pi/3) \end{bmatrix}$$

- Which yields

$$[\mathbf{v}_{qd0}] = V_m\begin{bmatrix} \cos(\theta_q - \omega t) \\ \sin(\theta_q - \omega t) \\ 0 \end{bmatrix}$$

$$\theta_q = \omega t + \theta_0'$$

$$[\mathbf{v}_{qd0}] = V_m\begin{bmatrix} \cos\theta_0' \\ \sin\theta_0' \\ 0 \end{bmatrix}$$

Power Transfer of Park's Transformation

- The power in *abc* reference frame is expressed as

$$P_{abc} = [\mathbf{v}_{abc}]^T [\mathbf{i}_{abc}] \quad \text{where} \quad [\mathbf{v}_{abc}] = \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} \quad \text{and} \quad [\mathbf{i}_{abc}] = \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix}$$

- Using inverse Park's transformation on the voltage and current yields:

$$[\mathbf{v}_{abc}] = [\mathbf{T}_{qd0}]^{-1} [\mathbf{v}_{qd0}]$$

$$[\mathbf{i}_{abc}] = [\mathbf{T}_{qd0}]^{-1} [\mathbf{i}_{qd0}]$$

$$P_{abc} = \left([\mathbf{T}_{qd0}]^{-1} [\mathbf{v}_{qd0}]\right)^T \left([\mathbf{T}_{qd0}]^{-1} [\mathbf{i}_{qd0}]\right)$$

Power Transfer of Park's Transformation

$$P_{abc} = \left(\left[\mathbf{T}_{qd0}\right]^{-1}\left[\mathbf{v}_{qd0}\right]\right)^{T}\left(\left[\mathbf{T}_{qd0}\right]^{-1}\left[\mathbf{i}_{qd0}\right]\right)$$

$$\Rightarrow P_{abc} = \left[\mathbf{v}_{qd0}\right]^{T}\left(\left[\mathbf{T}_{qd0}\right]^{-1}\right)^{T}\left[\mathbf{T}_{qd0}\right]^{-1}\left[\mathbf{i}_{qd0}\right]$$

- Using the inverse transformation matrix and its transpose we have:

$$\left(\left[\mathbf{T}_{qd0}\right]^{-1}\right)^{T}\left[\mathbf{T}_{qd0}\right]^{-1} = \begin{bmatrix} \frac{3}{2} & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & 3 \end{bmatrix} \Rightarrow P_{abc} \neq P_{qd0}$$

- Therefore Park's transformation is **not power-invariant**.
- To have power-invariant property the above matrix should be identity.

Generalized Park's Transformation

- The rotational velocity of the d-q frame can be **arbitrary** (synchronous, asynchronous or zero)

$$\left[\mathbf{f}_{qd0}\right] = \left[\mathbf{T}_{qd0}(\theta)\right]\left[\mathbf{f}_{abc}\right]$$

where

$$\left[\mathbf{T}_{qd0}(\theta)\right] = \frac{2}{3}\begin{bmatrix} \cos\theta & \cos(\theta - 2\pi/3) & \cos(\theta + 2\pi/3) \\ \sin\theta & \sin(\theta - 2\pi/3) & \sin(\theta + 2\pi/3) \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$