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Chapter 3
Principle of Electromechanical Energy Conversion

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Electromechanical Relations
Motoring Case

\[ P_{elec} \rightarrow \text{Electrical System} \rightarrow \text{Coupling Field} \rightarrow \text{Mechanical System} \rightarrow P_{mech} \]

- Electrical System
  - Electrical losses
    - Ohmic (copper) losses

- Coupling Field
  - Magnetic losses
    - Hysteresis losses
    - Eddy current losses

- Mechanical System
  - Mechanical losses
    - Bearing losses
    - Ventilation
      - Windage

\[ W_{elec} = W_{field} + W_{mech} \]
\[ dW_{elec} = dW_{field} + dW_{mech} \]
Magnetic System with Single Excitation

\[ e = v - ri \]

\[ ei = vi - ri^2 \]

\[ eidt = vidt - ri^2 dt \]

\[ dW_{elec} = dW_{field} + dW_{mech} \]

\[ dW_{elec} = dW_{field} = eidt \]

\[ dW_{field} = id\lambda \]

\[ dW_{mech} = 0 \]

\[ e = \frac{d\lambda}{dt} \]

\[ \lambda = N\phi \]
Magnetic Relay with Single Excitation

Assumption 1: The movable part cannot move or is not allowed to move

\[ dW_{mech} = 0 \]

\[ dW_{elec} = dW_{field} = id\lambda \]
Magnetic Relay with Single Excitation

**Assumption 1:** The movable part cannot move

\[
\begin{align*}
    dW_{\text{mech}} &= 0 \\
    dW_{\text{elec}} &= dW_{\text{field}} = id\lambda \\
    W_{\text{field}} &= \int_0^\lambda id\lambda \\
    \lambda &= N\phi \\
    Ni &= H_c l_c + H_g l_g \\
    H_g &= \frac{B}{\mu_0} \\
    W_{\text{field}} &= \int_0^B \left( H_c l_c + \frac{B}{\mu_0} l_g \right) AdB
\end{align*}
\]
Magnetic Relay with Single Excitation

Assumption 1: The movable part cannot move

\[ W_{\text{field}} = \int_0^B \left( H_c l_c + \frac{B}{\mu_0} l_g \right) A dB \]

\[ W_{\text{field}} = V_c \int_0^B H_c dB + V_g \frac{B^2}{2\mu_0} \]

- Stored magnetic energy in the core
- Stored magnetic energy in the air-gap

where

- \( V_c \) is the core volume
- \( V_g \) is the air-gap volume
Magnetic Relay with Single Excitation

Assumption 1: The movable part cannot move

In the case of linear systems:
\( \mu_r \rightarrow \text{constant} \)

\[ H_c = \frac{B}{\mu_0 \mu_r} \]

\[ W_{field} = V_c \frac{B^2}{2\mu_0 \mu_r} + V_g \frac{B^2}{2\mu_0} \]

Stored magnetic energy in the core

Stored magnetic energy in the air-gap
Magnetic Relay with Single Excitation

Example 1: In the following system if the air-gap flux density is 1 T and air-gap length is constant and fringing effect is neglected, calculate:

a) The DC source voltage;
b) Stored magnetic energy.
Magnetic Relay with Single Excitation

Solution 1: \( B_g = B_c = 1 \text{T} \)

From the curve \( H_c = 670 \text{ At/m} \)

\[
H_g = \frac{B_g}{\mu_0} = \frac{1}{4\pi \times 10^{-7}} = 795 \times 10^3 \text{ At/m}
\]

\( l_c = 0.6 \text{ m} \quad l_g = 5 \text{ mm} \)

\[
N_i = H_c l_c + 2H_g l_g
\]

\[
i = \frac{1}{N} \left( H_c l_c + 2H_g l_g \right) = 33.4 \text{ A}
\]

\( v_{dc} = ri = 167 \text{ V} \)
Magnetic Relay with Single Excitation

Solution 1: \[ B_g = B_c = 1 \text{ T} \]

\[
W_{\text{field}} = V_c \int_0^1 H_c dB + V_g \frac{B^2}{2\mu_0}
\]

\[ V_g = 2 \times 0.005 \times 0.1 \times 0.05 = 5 \times 10^{-5} \text{ m}^3 \]

\[ V_c = 0.6 \times 0.1 \times 0.05 = 3 \times 10^{-3} \text{ m}^3 \]

\[
W_{\text{field}} = 3 \times 10^{-3} \times 335 + 5 \times 10^{-5} \times \frac{1^2}{2 \times 4\pi \times 10^{-7}}
\]

\[ W_{\text{field}} = 1.005 + 19.895 = 20.9 \text{ J} \]
Energy and Coenergy

Energy

\[ W_{field} = \int_0^\lambda id\lambda \]

Coenergy

\[ W'_{field} = \int_0^i \lambda di \]

\[ W_{field} + W'_{field} = \lambda i \]
The $\lambda$-$i$ curve varies with the air-gap length.

Increasing air-gap length
Magnetic Relay with Single Excitation

**Assumption 2:** The movable part can move but **slowly**

In this case the **current remains constant** during the movement.
Magnetic Relay with Single Excitation

**Assumption 2:** The movable part can move but **slowly**

From $o$ to $a$: No movement yet, therefore

$$dW_{elec} = dW_{field} = id\lambda = A_{oad}$$

$$dW_{mech} = 0$$
Magnetic Relay with Single Excitation

Assumption 2: The movable part can move but slowly

From a to b:  
\[ dW_{elec} = id\lambda = i_1(\lambda_2 - \lambda_1) = A_{abcd} \]

\[ dW_{field} = W_{field(b)} - W_{field(a)} = A_{abc} - A_{oad} \]

\[ dW_{elec} = dW_{field} + dW_{mech} \]

\[ dW_{mech} = A_{oab} \]
Magnetic Relay with Single Excitation

Assumption 3: The movable part can move but very fast

In this case the flux linkage remains constant during the movement.
Magnetic Relay with Single Excitation

Assumption 3: The movable part can move but very fast

From o to a: No movement yet, therefore

\[ dW_{elec} = dW_{field} = id\lambda = A_{oad} \]

\[ dW_{mech} = 0 \]
Magnetic Relay with Single Excitation

Assumption 3: The movable part can move but very fast

From \( a \) to \( c \):
\[
dW_{elec} = id\lambda = 0
\]

\[
dW_{field} = W_{field(c)} - W_{field(a)} = A_{ocd} - A_{oad}
\]

\[
dW_{elec} = dW_{field} + dW_{mech}
\]
Magnetic Relay with Single Excitation

Assumption 3: The movable part can move but very fast

From c to b: No movement, therefore \( dW_{mech} = 0 \)

\[
dW_{elec} = dW_{field} = id\lambda = A_{cbed}
\]
Some Questions (H.W)

1- Why does the current remain constant during the slow movement?

2- Why does the flux linkage remain constant during the fast movement?

3- Why should the currents at the open and closed stages be the same?
Assumption 4: The movable part moves with normal speed
Magnetic Relay with Single Excitation

**Assumption 4:** The movable part moves with *normal speed*

From *o* to *a*: No movement yet, therefore

\[ dW_{elec} = dW_{field} = id\lambda = A_{oad} \]
Magnetic Relay with Single Excitation

Assumption 4: The movable part moves with normal speed

From \( a \) to \( c \):

\[
dW_{elec} = id\lambda = A_{aced}
\]

\[
dW_{field} = W_{field(c)} - W_{field(a)} = A_{oce} - A_{oad}
\]

\[
dW_{elec} = dW_{field} + dW_{mech}
\]

\[
dW_{mech} = A_{oac}
\]
Assumption 4: The movable part moves with normal speed

From c to b: No Movement, therefore

\[ dW_{elec} = dW_{field} = id\lambda = A_{cbfe} \]
Example 2: The energy conversion cycles of two machines are OABO and OABCO curves shown below. If the energy conversion efficiency is defined as follows, calculate $R_1$ and $R_2$

\[
R = \frac{\text{Converted Energy}}{\text{Input Electrical Energy}}
\]

\[
R_1 = \frac{\int_{\lambda_1}^{\lambda_2} E_1(\lambda) \, d\lambda}{\int_{\lambda_1}^{\lambda_2} E_{in}(\lambda) \, d\lambda}
\]

\[
R_2 = \frac{\int_{\lambda_1}^{\lambda_2} E_2(\lambda) \, d\lambda}{\int_{\lambda_1}^{\lambda_2} E_{in}(\lambda) \, d\lambda}
\]
Electromechanical Energy Conversion

Solution 2: Part (1)

\[ W_{elec1} = \int_{\lambda_A}^{\lambda_B} i d\lambda + \int_{\lambda_B}^{\lambda_O} i d\lambda = \int_{0}^{1} 4 \lambda d\lambda + \int_{1}^{3} 4 d\lambda = 10 \]

\[ W_{mech1} = A_{OABO} = 4 \]

\[ R_1 = \frac{W_{mech1}}{W_{elec1}} = \frac{4}{10} = 0.4 \]
Electromechanical Energy Conversion

Solution 2: Part (2)

\[ W_{elec2} = \int_{\lambda_O}^{\lambda_A} id\lambda + \int_{\lambda_A}^{\lambda_B} id\lambda + \int_{\lambda_B}^{\lambda_C} id\lambda = \int_0^1 4\lambda d\lambda + \int_1^3 4d\lambda \int_3^3 id\lambda = 10 \]

\[ W_{mech2} = A_{OABCO} = 7 \]

\[ R_2 = \frac{W_{mech2}}{W_{elec2}} = \frac{7}{10} = 0.7 \]
Mechanical Force

The average force is calculated as the mechanical work divided by the displacement

\[ F_{ave} = \frac{\text{Mechanical Work}}{\text{Displacement}} \]

\[ \lambda \quad (\text{Wb.t}) \]

\[ x = g \quad \text{Fully closed} \]

\[ x + dx \quad x = 0 \quad \text{Fully open} \]

(no movement yet)
Mechanical Force

\[ dW_{\text{mech}} = A_{oab} \]

If the flux linkage is constant (fast movement),

\[ dW_{\text{mech}} = A_{oah} \]

\[ dW_{\text{elec}} = 0 \]

\[ dW_{\text{mech}} = -dW_{\text{field}} \]

\[ F \, dx = -dW_{\text{field}} \]

\[ F = -\frac{\partial W_{\text{field}}(\lambda, x)}{\partial x} \quad \lambda = \text{cte} \]
Mechanical Force

\[ dW_{\text{mech}} = A_{oab} \]

- If the current is constant (slow movement),
  \[ dW_{\text{mech}} = A_{oac} \]

\[ dW_{\text{elec}} = A_{acdf} = A_{okcd} - A_{okaf} \]

\[ dW_{\text{elec}} = (W_{\text{field}(c)} + W'_{\text{field}(c)}) - (W_{\text{field}(a)} + W'_{\text{field}(a)}) = dW_{\text{field}} + dW'_{\text{field}} \]

\[ dW_{\text{elec}} = dW_{\text{field}} + dW_{\text{mech}} \]

\[ Fdx = dW'_{\text{field}} \]

\[ F = \frac{\partial W'_{\text{field}}(i, x)}{\partial x} \]

\[ i = \text{cte} \]
## Force and Torque Calculation

<table>
<thead>
<tr>
<th>Translational movement</th>
<th>Rotational movement</th>
</tr>
</thead>
<tbody>
<tr>
<td>Force</td>
<td>Torque</td>
</tr>
<tr>
<td>( F = -\frac{\partial W_{\text{field}}(\lambda, x)}{\partial x} \bigg</td>
<td>_{\lambda=\text{cte}} )</td>
</tr>
<tr>
<td>( F = \frac{\partial W'_{\text{field}}(i, x)}{\partial x} \bigg</td>
<td>_{i=\text{cte}} )</td>
</tr>
</tbody>
</table>

where \( \theta \) is the angular position.
Electromechanical Energy Conversion

**Example 3:** In an electromechanical system \( i = \left( \lambda x / 0.09 \right)^2 \) for \( 0 \leq i \leq 4 \) and \( 3 \leq x \leq 10 \text{ cm} \). Calculate the force exerted on the movable part if the current is 3 A and air-gap length is 5 cm.

**Solution:** 1\textsuperscript{st} method (Coenergy)

\[
W'_{\text{field}} = \int_0^i \lambda di
\]

\[
F = \left. \frac{\partial W'_{\text{field}}(i, x)}{\partial x} \right|_{i=\text{cte}}
\]

2\textsuperscript{nd} method (Energy)

\[
W_{\text{field}} = \int_0^\lambda id\lambda
\]

\[
F = -\left. \frac{\partial W_{\text{field}}(\lambda, x)}{\partial x} \right|_{\lambda=\text{cte}}
\]
Electromechanical Energy Conversion

\[ i = \left( \frac{\lambda x}{0.09} \right)^2 \quad \text{for} \quad 0 \leq i \leq 4 \quad 3 \leq x \leq 10 \text{ cm} \]

\[ i = 3 \text{ A} \quad x = 5 \text{ cm} \quad F = ? \]

**Solution 3: 1st method (Coenergy)**

\[ \lambda = \frac{0.09 \sqrt{i}}{x} \quad \Rightarrow \quad W'_{\text{field}} = \int_0^i \lambda di = \int_0^i \frac{0.09 \sqrt{i}}{x} di = \frac{0.09}{x} \frac{2}{3} i^{3/2} \]

\[ F = \frac{\partial W'_{\text{field}}(i, x)}{\partial x} \bigg|_{i=\text{cte}} = -\frac{0.09}{x^2} \frac{2}{3} i^{3/2} \]

For \( i = 3 \text{ A} \) and \( x = 5 \text{ cm} \)

\[ F = -\frac{0.09}{x^2} \frac{2}{3} i^{3/2} = -124 \text{ N} \]
Electromechanical Energy Conversion

\[ i = \left( \frac{\lambda x}{0.09} \right)^2 \quad 0 \leq i \leq 4 \quad 3 \leq x \leq 10 \text{ cm} \]

\[ i = 3 \text{ A} \quad x = 5 \text{ cm} \quad F = ? \]

**Solution 3: 2\textsuperscript{nd} method (Energy)**

\[
W_{\text{field}} = \int_0^\lambda i d\lambda = \int_0^\lambda \left( \frac{\lambda x}{0.09} \right)^2 d\lambda = \frac{x^2}{0.09^2} \frac{\lambda^3}{3}
\]

\[
F = -\left. \frac{\partial W_{\text{field}}(\lambda, x)}{\partial x} \right|_{\lambda=\text{cte}} = -\frac{2x}{0.09^2} \frac{\lambda^3}{3} = -\frac{2x}{0.09^2} \frac{1}{3} \left( \frac{0.09 \sqrt{i}}{x} \right)^3
\]

For \( i = 3 \text{ A} \) and \( x = 5 \text{ cm} \)

\[
F = -\frac{2x}{0.09^2} \frac{1}{3} \left( \frac{0.09 \sqrt{i}}{x} \right)^3 = -124.7 \text{ N}
\]
Electromechanical Energy Conversion

Example 4: In the following system assume flux is constant during the movement, calculate the average force.
Solution 4: Since the permeability of the core goes to infinity, the system is linear.

\[ \lambda = Li \quad \lambda = N\phi \]

\[ L = \frac{N^2}{\mathcal{R}} \]

\[ W_{\text{field}} = \frac{1}{2}Lt^2 = \frac{1}{2} \lambda i = \frac{1}{2} \frac{\lambda^2}{L} = \frac{1}{2} \mathcal{R}\phi^2 \]

\[ W_{\text{field}} = \frac{1}{2} \left( \frac{g - x}{\mu_0 A} \right) \phi^2 \]

\[ F = -\frac{\partial W_{\text{field}}(\lambda, x)}{\partial x} \bigg|_{\lambda=\text{cte}} = \frac{1}{2} \phi^2 \frac{1}{\mu_0 A} \]
Electromechanical Energy Conversion

Example 5: In an electromechanical system the $\lambda - i$ characteristics is defined as $i = \lambda^{3/2} + 2.5\lambda(x-1)^2$ for $0 < x < 1^m$, calculate the average force when $x = 0.6^m$.

$$W_{field} = \int_0^\lambda i d\lambda = \int_0^\lambda (\lambda^{3/2} + 2.5\lambda(x-1)^2) d\lambda$$

$$W_{field} = \frac{2}{5} \lambda^{5/2} + \frac{5}{4} \lambda^2 (x-1)^2$$

$$F = -\frac{\partial W_{field}(\lambda, x)}{\partial x} \bigg|_{\lambda=cte} = -\frac{5}{4} 2\lambda^2 (x - 1)$$

$$F(x = 0.6) = \lambda^2$$
Developed Torque in Doubly Excited Systems

Consider the following electric motor with double excitation.

\[
dW_{elec} = i_s \, d\lambda_s + i_r \, d\lambda_r
\]

where

\[
\begin{align*}
\lambda_s &= L_s i_s + M_{sr} i_r \\
\lambda_r &= M_{sr} i_s + L_r i_r
\end{align*}
\]

- \( i_s \) stator current
- \( i_r \) rotor current
- \( \lambda_s \) stator flux linkage
- \( \lambda_r \) rotor flux linkage
- \( L_s \) stator self-inductance
- \( L_r \) rotor self-inductance
- \( M_{sr} \) mutual inductance
Developed Torque in Doubly Excited Systems

The inductances are defined as follows

\[ L_s = \frac{N_s^2}{\mathcal{R}_s} \]
\[ L_r = \frac{N_r^2}{\mathcal{R}_r} \]
\[ M_{sr} = \frac{N_s N_r}{\mathcal{R}_{sr}} \]

where

- \( N_s \) stator number of turns
- \( N_r \) rotor number of turns
- \( \mathcal{R}_s \) reluctance seen by stator flux
- \( \mathcal{R}_r \) reluctance seen by rotor flux
- \( \mathcal{R}_{sr} \) reluctance seen by resultant flux
Developed Torque in Doubly Excited Systems

Assumption 1: The rotor cannot rotate

\[ dW_{elec} = i_s d(L_s i_s + M_{sr} i_r) + i_r d(M_{sr} i_s + L_r i_r) \]

\[ dW_{elec} = L_s i_s di_s + M_{sr} i_s di_r + M_{sr} i_r di_s + L_r i_r di_r \]

\[ dW_{elec} = dW_{field} + dW_{mech} \]

\[ dW_{field} = L_s i_s di_s + M_{sr} d(i_s i_r) + L_r i_r di_r \]

\[ W_{field} = L_s \int_0^i s i_s di_s + M_{sr} \int_0^i s i_r d(i_s i_r) + L_r \int_0^r i_r di_r \]

\[ W_{field} = \frac{1}{2} L_s i_s^2 + M_{sr} i_s i_r + \frac{1}{2} L_r i_r^2 \]  

(1)
Developed Torque in Doubly Excited Systems

Assumption 2: The rotor can rotate

\[ dW_{elec} = i_s d(L_s i_s + M_{sr} i_r) + i_r d(M_{sr} i_s + L_r i_r) \]

\[ dW_{elec} = L_s i_s d i_s + i_s^2 d L_s + M_{sr} i_s d i_r + i_r i_s d M_{sr} \]
\[ + M_{sr} i_r d i_s + i_s i_r d M_{sr} + L_r i_r d i_r + i_r^2 d L_r \]

Taking differentiation from the relation \( W_{field} = \frac{1}{2} L_s i_s^2 + M_{sr} i_s i_r + \frac{1}{2} L_r i_r^2 \)

yields

\[ dW_{field} = L_s i_s d i_s + \frac{1}{2} i_s^2 d L_s + M_{sr} i_s d i_r + i_r i_s d M_{sr} \]
\[ + M_{sr} i_r d i_s + L_r i_r d i_r + \frac{1}{2} i_r^2 d L_r \]
Developed Torque in Doubly Excited Systems

Assumption 2: The rotor can rotate

\[ dW_{elec} = L_s i_s d_i_s + i_s^2 dL_s + M_{sr} i_s d_i_r + i_r i_s dM_{sr} \]
\[ + M_{sr} i_r d_i_s + i_s i_r dM_{sr} + L_r i_r d_i_r + i_r^2 dL_r \]

\[ dW_{field} = L_s i_s d_i_s + \frac{1}{2} i_s^2 dL_s + M_{sr} i_s d_i_r + i_r i_s dM_{sr} \]
\[ + M_{sr} i_r d_i_s + L_r i_r d_i_r + \frac{1}{2} i_r^2 dL_r \]

\[ dW_{elec} = dW_{field} + dW_{mech} \]

\[ dW_{mech} = \frac{1}{2} i_s^2 dL_s + i_s i_r dM_{sr} + \frac{1}{2} i_r^2 dL_r \]
Developed Torque in Doubly Excited Systems

**Assumption 2:** The rotor can rotate

\[ dW_{\text{mech}} = \frac{1}{2} i_s^2 dL_s + i_s i_r dM_{sr} + \frac{1}{2} i_r^2 dL_r \]

Since the torque is defined as

\[ T = \frac{dW_{\text{mech}}}{d\theta} \]

it yields

\[ T = \frac{1}{2} i_s^2 \frac{dL_s}{d\theta} + i_s i_r \frac{dM_{sr}}{d\theta} + \frac{1}{2} i_r^2 \frac{dL_r}{d\theta} \]

**Reluctance torque**

**Electromagnetic torque**

**Reluctance torque**
Developed Torque in Doubly Excited Systems

**Case 1:** Both rotor and stator have salient structures:

\[ L_s, L_r \text{ and } M_{sr} \text{ are function of } \theta. \]

\[
T = \frac{1}{2} i_s^2 \frac{dL_s}{d\theta} + i_s i_r \frac{dM_{sr}}{d\theta} + \frac{1}{2} i_r^2 \frac{dL_r}{d\theta}
\]

**Electromagnetic torque**

- Reluctance torque due to rotor saliency
- Reluctance torque due to stator saliency

Reluctance torque is independent of **current direction**.
Developed Torque in Doubly Excited Systems

Case 2: Rotor has salient structures but stator is cylindrical:

\[ L_s \text{ and } M_{sr} \text{ are function of } \theta. \]

\[
T = \frac{1}{2} i_s^2 \frac{dL_s}{d\theta} + i_s i_r \frac{dM_{sr}}{d\theta} + \frac{1}{2} i_r^2 \frac{dL_r}{d\theta}
\]

- Electromagnetic torque
- Reluctance torque due to rotor saliency
Developed Torque in Doubly Excited Systems

Case 3: Stator has salient structures but rotor is cylindrical:

\[ L_r \] and \[ M_{sr} \] are function of \( \theta \).

\[
T = \frac{1}{2} i_s^2 \frac{dL_s}{d\theta} + i_s i_r \frac{dM_{sr}}{d\theta} + \frac{1}{2} i_r^2 \frac{dL_r}{d\theta}
\]

Electromagnetic torque

Reluctance torque due to rotor saliency
Developed Torque in Doubly Excited Systems

Case 4: Both rotor and stator are cylindrical (non-salient):
only \( M_{sr} \) is a function of \( \theta \).

\[
T = \frac{1}{2} i_s^2 \frac{dL_s}{d\theta} + i_s i_r \frac{dM_{sr}}{d\theta} + \frac{1}{2} i_r^2 \frac{dL_r}{d\theta}
\]

Electromagnetic torque

\[
T = i_s i_r \frac{dM_{sr}}{d\theta}
\]
Developed Torque in Doubly Excited Systems

Example 6: In the following figure the rotor has no winding and the self-inductance is assumed to be \( L_s = L_0 + L_1 \cos 2\theta \) where \( \theta \) is the rotor position. If stator current is \( i_s = I_m \sin \omega t \).

a) Calculate the torque exerted on the rotor.

b) Find the conditions in which the average torque is not zero if \( \theta = \omega_m t + \delta \) where \( \omega_m \) is the angular velocity of the rotor and \( \delta \) is the initial position of the rotor.

\[
T = \frac{1}{2} i_s^2 \frac{dL_s}{d\theta} + i_s i_r \frac{dM_{sr}}{d\theta} + \frac{1}{2} i_r^2 \frac{dL_r}{d\theta}
\]
Developed Torque in Doubly Excited Systems

Solution 6: \[ L_s = L_0 + L_1 \cos 2\theta \quad i_s = I_m \sin \omega t \quad i_r = 0 \]

Part a)

\[ T = \frac{1}{2} i_s^2 \frac{dL_s}{d\theta} + i_s i_r \frac{dM_{sr}}{d\theta} + \frac{1}{2} i_r^2 \frac{dL_r}{d\theta} \]

Since rotor has no winding:

\[ T = \frac{1}{2} i_s^2 \frac{dL_s}{d\theta} \]

\[ T = \frac{1}{2} (I_m \sin \omega t)^2 \frac{d}{d\theta} (L_0 + L_1 \cos 2\theta) \]

\[ T = -I_m^2 L_1 \sin 2\theta \sin^2 \omega t \]
Developed Torque in Doubly Excited Systems

Solution 6:  \[ L_s = L_0 + L_1 \cos 2\theta \]

Part b)
\[ \theta = \omega_m t + \delta \]
\[ i_s = I_m \sin \omega t \]
\[ i_r = 0 \]

\[ T = -I_m^2 L_1 \sin 2\theta \sin^2 \omega t \]

\[ T = -I_m^2 L_1 \sin 2(\omega_m t + \delta) \frac{1-\cos 2\omega t}{2} \]

\[ T = -\frac{1}{2} I_m^2 L_1 \{ \sin 2(\omega_m t + \delta) - \frac{1}{2} \sin 2[(\omega_m + \omega)t + \delta] - \frac{1}{2} \sin 2[(\omega_m - \omega)t + \delta] \} \]

To have non-zero average torque the coefficient of \( t \) in one of the above sin terms should be zero:

\[ \omega_m = 0 \quad \text{or} \quad \omega_m = -\omega \quad \text{or} \quad \omega_m = \omega \]
Developed Torque in Doubly Excited Systems

Solution 6:
Part b)

\[ T = -\frac{1}{2} I_m^2 L_1 \left\{ \sin 2(\omega_m t + \delta) - \frac{1}{2} \sin 2[(\omega_m + \omega)t + \delta] - \frac{1}{2} \sin 2[(\omega_m - \omega)t + \delta] \right\} \]

1) If \( \omega_m = 0 \)

\[ T_{ave} = -\frac{1}{2} I_m^2 L_1 \sin 2 \delta \]

2) If \( \omega_m = \pm \omega \)

\[ T_{ave} = \frac{1}{4} I_m^2 L_1 \sin 2 \delta \]
**Example 7:** In the following system assume the movable part is fixed and the air-gap length is $d = 1^\text{mm}$.

1) Calculate the current and voltage if flux density in air-gap is 0.5 T.
2) Calculate the stored energy.
3) Obtain the force exerted on the movable part.
4) Compute the inductance of the winding.

![Diagram of an electromechanical system with a movable part, a spring, and flux density graph.](image-url)
Solution 7: \[ d = l_g = 1 \text{ mm} \quad l_c = 0.6 \text{ m} \quad \mu_0 = 4\pi \times 10^{-7} \]

Part 1) \[ i = ? \quad V = ? \]

\[ B_g = B_c = 0.5 \text{ T} \]

From the curve \[ H_c = 350 \text{ At/m} \]

\[ i = \frac{H_c l_c + H_g l_g}{N} = \frac{H_c l_c + \frac{B_g}{\mu_0} l_g}{N} \]

\[ i = 1.215 \text{ A} \]

\[ V = r i = 4.86 \text{ V} \]
Solution 7: \( d = l_g = 1 \text{mm} \)

Part 2) \( W_{\text{field}} = ? \)

\[
W_{\text{field}} = V_c \int_0^B H_c dB + V_g \frac{B^2}{2\mu_0}
\]

\( W_{\text{field}} = 0.38 \text{ J} \)

\( l_c = 0.6 \text{ m} \)
\( \mu_0 = 4\pi \times 10^{-7} \)
\( B = B_g = B_c = 0.5 \text{ T} \)
\( H_c = 350 \text{ At/m} \)
\( V_c = 0.6 \times 0.05 \times 0.05 = 1.5 \times 10^{-3} \text{ m}^3 \)
\( V_g = 0.001 \times 0.05 \times 0.05 = 2.5 \times 10^{-6} \text{ m}^3 \)
Electromechanical Energy Conversion

Solution 7: \( d = l_g = 1 \text{ mm} \)

Part 3) \( F = ? \)

\[
B = B_g = B_c = 0.5 \text{ T}
\]

\[
A_g = 0.05 \times 0.05 = 2.5 \times 10^{-3} \text{ m}^2
\]

\[
\mu_0 = 4\pi \times 10^{-7}
\]

\[
l_c = 0.6 \text{ m}
\]

\[
W_{\text{field}} = l_c A_c \int_{0}^{B} H_c dB + l_g A_g \frac{B^2}{2\mu_0}
\]

\[
l_g \rightarrow x
\]

\[
F = -\left. \frac{\partial W_{\text{field}}}{\partial x} \right|_{x=\text{cte}} = -A_g \frac{B^2}{2\mu_0}
\]

\[
F = -248.7 \text{ N}
\]
Electromechanical Energy Conversion

Solution 7: \( d = l_g = 1 \text{ mm} \)

Part 3) \( L = ? \)

\[
L = \frac{\lambda}{i} = \frac{N\phi}{i} = \frac{NAB}{i}
\]

\[
L = \frac{500 \times 2.5 \times 10^{-3} \times 0.5}{1.215}
\]

\( L = 0.514 \text{ H} \)

\( l_c = 0.6 \text{ m} \)

\( \mu_0 = 4\pi \times 10^{-7} \)

\( B = B_g = B_c = 0.5 \text{ T} \)

\( A_g = 0.05 \times 0.05 = 2.5 \times 10^{-3} \text{ m}^2 \)
Electromechanical Energy Conversion

Example 8: In the following system assume the current is 1.215 A and the air-gap is fully closed. \( d = 0 \)
1) Calculate the flux density.
2) Calculate the force exerted on spring.
3) Calculate the stored energy
Electromechanical Energy Conversion

Solution 8: \[ d = l_g = 0 \]

Part 1) \[ B = ? \]

\[ Ni = H_c l_c + H_g l_g \]

From the curve \[ B_c = 1.21 \, \text{T} \]

\[ H_c = \frac{Ni}{l_c} = 1012 \, \text{At/m} \]
Electromechanical Energy Conversion

Solution 8: \( d = l_g = 0 \)

Part 2) \( W_{field} = ? \)

\[
W_{field} = V_c \int_0^B H_c dB + V_g \frac{B^2}{2\mu_0}
\]

\[
W_{field} = V_c \int_0^B H_c dB
\]

\[ W_{field} = 0.92 \text{ J} \]
Solution 8: \( d = l_g = 0 \)

Part 3) \( F = ? \)

\[ B = B_g = B_c = 1.21 \, \text{T} \]

\[ A_g = 0.05 \times 0.05 = 2.5 \times 10^{-3} \, \text{m}^2 \]

\[ l_c = 0.6 \, \text{m} \]

\[ \mu_0 = 4\pi \times 10^{-7} \]

\[ W_{field} = l_c A_c \int_0^B H_c dB + l_g A_g \frac{B^2}{2\mu_0} \]

\[ l_g \rightarrow x \]

\[ F = -\frac{\partial W_{field}}{\partial x} \bigg|_{\lambda=\text{cte}} = -A_g \frac{B^2}{2\mu_0} \]

\[ F = -1456.4 \, \text{N} \]
Developed Torque

Example 9: In the following figure the rotor has no winding and the self-inductance is assumed to be $L_s = 0.1 - 0.3 \cos 2\theta - 0.2 \cos 4\theta$ where $\theta$ is the rotor position. If stator current is 10 Aac with frequency of 60 Hz:

a) In which rotational velocity the average torque is not zero if

$\theta = \omega_m t + \delta$ where $\omega_m$ is the angular velocity of the rotor and $\delta$ is the initial position of the rotor.

b) In the velocity obtained above calculate the maximum torque and mechanical power.

c) Obtain maximum torque at zero velocity.

\[
T = \frac{1}{2} i_s^2 \frac{dL_s}{d\theta} + i_s i_r \frac{dM_{sr}}{d\theta} + \frac{1}{2} i_r^2 \frac{dL_r}{d\theta}
\]
Developed Torque

Solution 9: \[ L_s = 0.1 - 0.3 \cos 2\theta - 0.2 \cos 4\theta \] \[ \theta = \omega_m t + \delta \]

Part a)
\[ i_s = 10 \cos \omega_s t \] \[ \omega_s = 2\pi \times 60 = 120\pi \]

\[ T = \frac{\frac{1}{2} i_s^2 dL_s}{d\theta} \]

\[ T = \frac{1}{2} (10 \cos \omega_s t)^2 \frac{d}{d\theta} (0.1 - 0.3 \cos 2\theta - 0.2 \cos 4\theta) \]

\[ T = 50 \cos^2 \omega_s t (0.6 \sin 2\theta + 0.8 \sin 4\theta) \]

\[ T = 50 \frac{1 + \cos 2\omega_s t}{2} [0.6 \sin 2(\omega_m t + \delta) + 0.8 \sin 4(\omega_m t + \delta)] \]
Developed Torque

Solution 9: \[ L_s = 0.1 - 0.3 \cos 2\theta - 0.2 \cos 4\theta \]

\[ \theta = \omega_m t + \delta \]

Part a)

\[ i_s = 10 \cos \omega_s t \]

\[ \omega_s = 2\pi \times 60 = 120\pi \]

\[ T = 50 \frac{1 + \cos 2\omega_s t}{2} \left[ 0.6 \sin 2(\omega_m t + \delta) + 0.8 \sin 4(\omega_m t + \delta) \right] \]

\[ \sin a \cos b = \frac{1}{2} [\sin(a + b) + \sin(a - b)] \]

\[ T = 25 \left\{ 0.6 \sin 2(\omega_m t + \delta) + 0.8 \sin 4(\omega_m t + \delta) \right. \\
+ 0.3 \sin[2(\omega_m + \omega_s)t + 2\delta] + 0.3 \sin[2(\omega_m - \omega_s)t + 2\delta] \\
+ 0.4 \sin[2(2\omega_m + \omega_s)t + 4\delta] + 0.4 \sin[2(2\omega_m - \omega_s)t + 4\delta] \right\} \]

\[ \omega_m = 0 \quad \text{or} \quad \omega_m = \pm \omega_s = \pm 120\pi \quad \text{or} \quad \omega_m = \pm \frac{1}{2} \omega_s = \pm 60\pi \]
Developed Torque

Solution 9: \( L_s = 0.1 - 0.3 \cos 2\theta - 0.2 \cos 4\theta \)

Part b)

\[ \omega_m = \pm \omega_s = \pm 120\pi \]

\[ T_{ave} = 7.5 \sin 2\delta \]

\[ T_{ave-max} = 7.5 \text{ Nm} \quad \text{at} \quad \delta = k\pi \pm \frac{\pi}{4} \]

\[ P = T_{ave-max} \omega_m = 7.5 \times 120\pi = 2827 \text{ W} \]

\[ \omega_m = \pm \frac{1}{2} \omega_s = \pm 60\pi \]

\[ T_{ave} = 10 \sin 4\delta \]

\[ T_{ave-max} = 10 \text{ Nm} \quad \text{at} \quad \delta = \frac{k\pi}{2} \pm \frac{\pi}{8} \]

\[ P = T_{ave-max} \omega_m = 10 \times 60\pi = 1885 \text{ W} \]
Developed Torque

**Solution 9:**

\[ L_s = 0.1 - 0.3 \cos 2\theta - 0.2 \cos 4\theta \]

Part c)

\[ \omega_m = 0 \quad T_{ave} = 15 \sin 2\delta + 20 \sin 4\delta \]

\[ \frac{\partial T_{ave}}{\partial \delta} = 0 \implies 30 \cos 2\delta + 80 \cos 4\delta = 0 \]

\[ 3 \cos 2\delta + 8 \left( 2 \cos^2 2\delta - 1 \right) = 0 \]

\[ 16 \cos^2 2\delta + 3 \cos 2\delta - 8 = 0 \]

\[ \begin{cases} \cos 2\delta = 0.62 \\ \cos 2\delta = -0.81 \end{cases} \]

\[ \delta = 25.8^\circ \]

\[ \delta = 71.9^\circ \]

\[ T_{ave-max} = 31 \text{ Nm} \]

\[ T_{ave-max} = -10 \text{ Nm} \]

\[ \theta = \omega_m t + \delta \]

\[ \omega_s = 2\pi \times 60 = 120\pi \]