In The Name of God The Most Compassionate, The Most Merciful

دانشگاه صنعتی شیراز

Linear Control Systems

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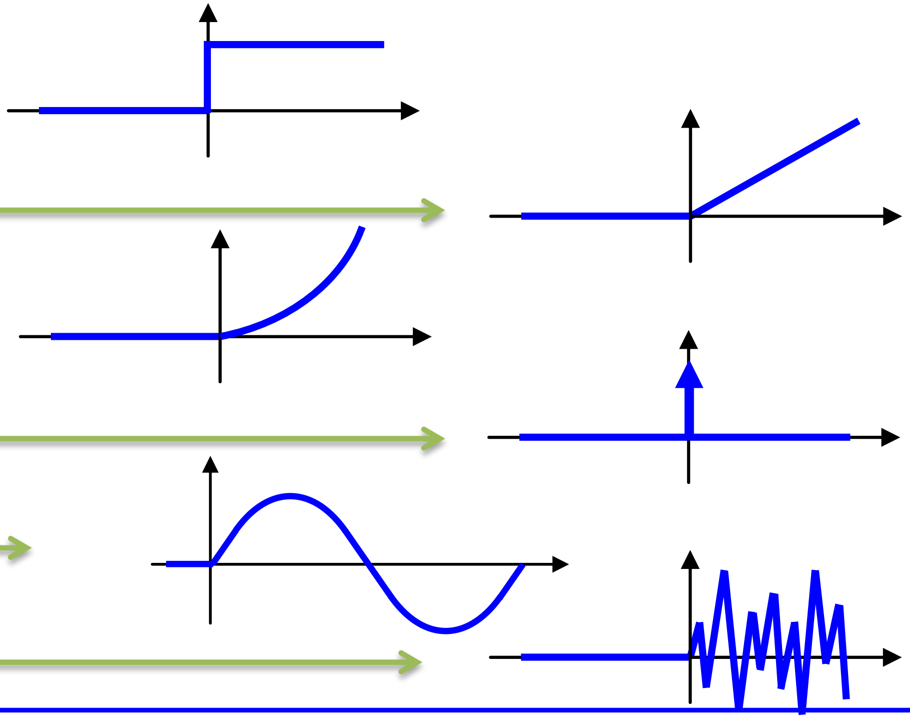
Introduction

- As mentioned, the first step in analyzing a control system was to derive a **mathematical model** of the system.

- Once such a model is obtained, various methods are available for the **analysis** of system performance.

- **Typical Test Signals**
  - Step functions
  - Ramp functions
  - acceleration functions
  - impulse functions
  - sinusoidal functions
  - White noise



Transient Response and Steady-State Response

- The **time response** of a control system consists of two parts:
  - the **transient** response and
  - the **steady-state** response.

- **Transient response** means the part of response goes from the initial state to the final state.

- By **steady-state response**, we mean the manner in which the system output behaves as t approaches infinity.

- Thus the system response $c(t)$ may be written as

$$c(t) = c_{tr}(t) + c_{ss}(t)$$

Stability of LTI Systems

- A control system is in **equilibrium** if, in the absence of any disturbance or input, the output stays in the same state.
- A linear time-invariant control system is **stable** if the output eventually comes back to its equilibrium state when the system is subjected to an initial condition.
- A linear time-invariant control system is **critically stable** if oscillations of the output continue forever.
- It is **unstable** if the output diverges without bound from its equilibrium state when the system is subjected to an initial condition.

First-Order System

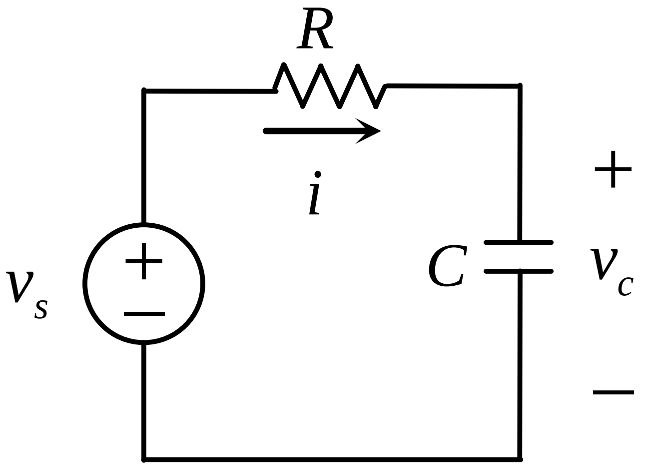
Consider the first-order system shown below. Physically, this system may represent an *RC* circuit.

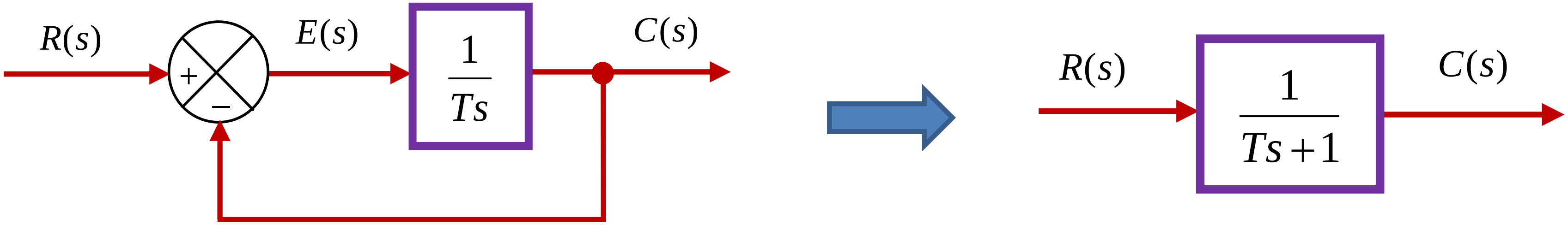
$$\frac{C(s)}{R(s)}=\frac{1}{Ts+1}$$

$$C(s)=V_c(s)$$

$$R(s)=V_s(s)$$

$$T=RC$$

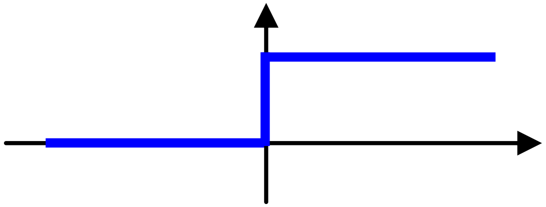




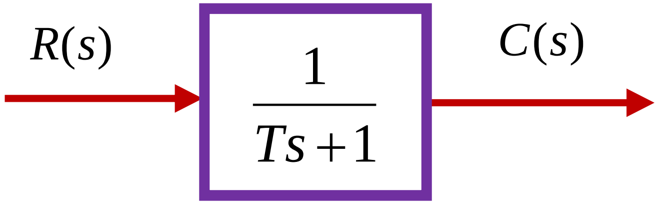
Unit-Step Response of First-Order Systems

If the input is a unit-step

$$R(s)=\frac{1}{s}$$



The output is calculated as

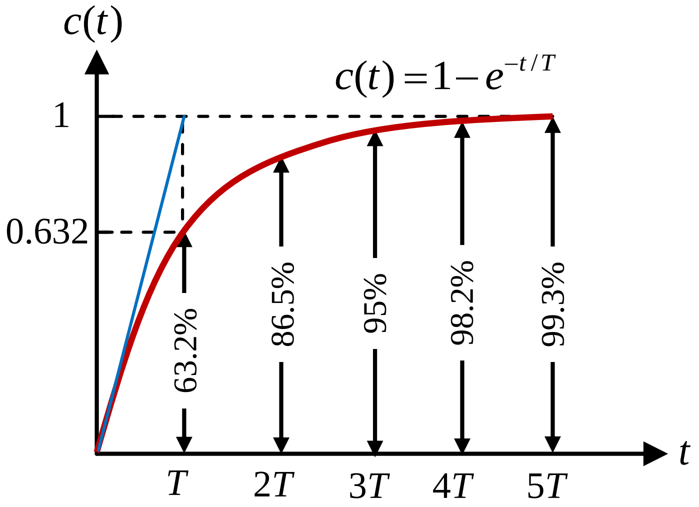


$$C(s)=\frac{1}{Ts+1}\frac{1}{s}$$

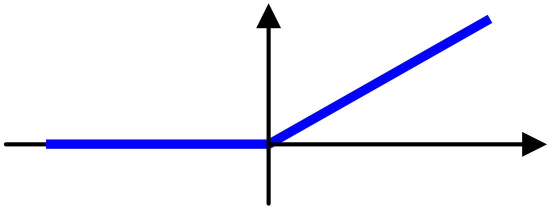
$$C(s)=\frac{1}{s}-\frac{T}{Ts+1}=\frac{1}{s}-\frac{1}{s+\frac{1}{T}}$$

Taking inverse Laplace transform

$$c(t)=1-e^{-t/T} \quad \text{for} \quad t\geq 0$$



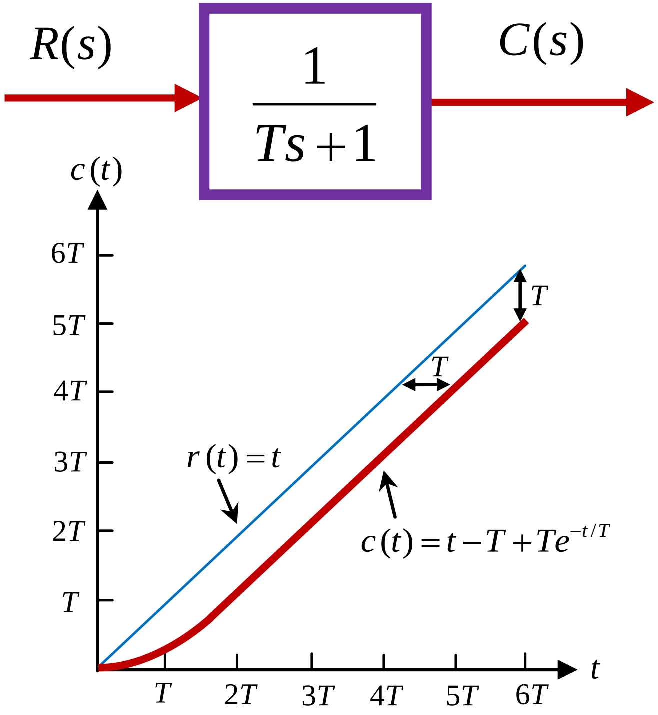
Unit-Ramp Response of First-Order Systems



If the input is a unit-ramp

$$R(s)=\frac{1}{s^2}$$

The output is calculated as



$$C(s)=\frac{1}{Ts+1}\frac{1}{s^2}$$

$$C(s)=\frac{1}{s^2}-\frac{T}{s}+\frac{T^2}{Ts+1}$$

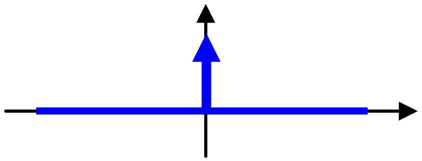
Taking inverse Laplace transform

$$c(t)=t-T+Te^{-t/T} \quad \text{for} \quad t\geq 0$$

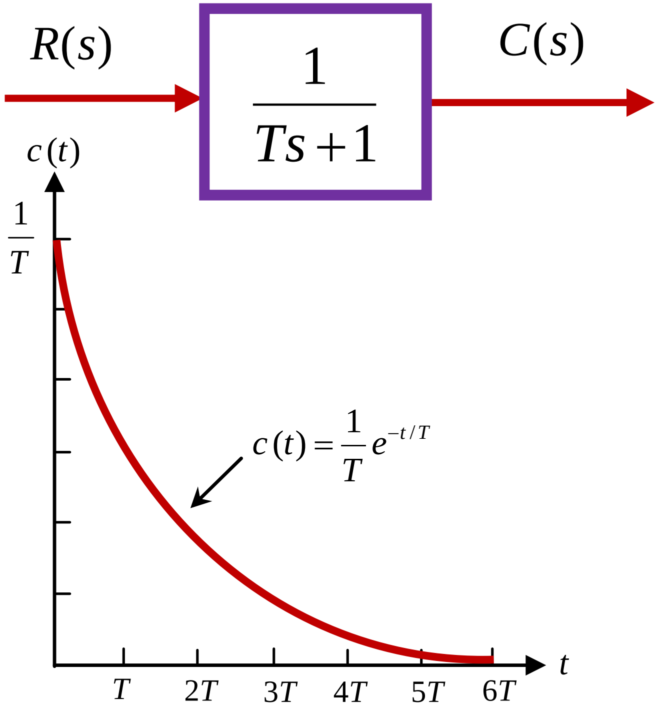
Unit-Impulse Response of First-Order Systems

If the input is a unit-ramp

$$R(s) = 1$$



The output is calculated as



$$C(s) = \frac{1}{Ts+1}$$

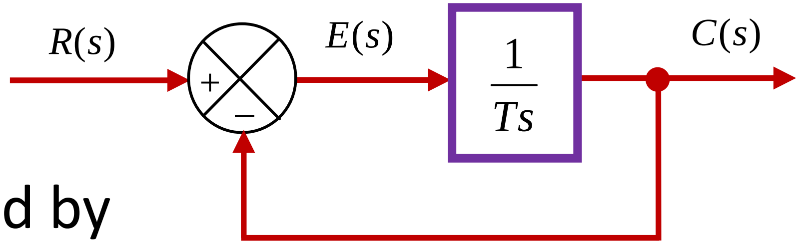
Taking inverse Laplace transform

$$c(t) = \frac{1}{T}e^{-t/T} \quad \text{for} \quad t \geq 0$$

Steady-State Error
First-Order Systems

Assume that the error is defined as

$$e(t) = r(t) - c(t)$$



And the steady-state error is calculated by

$$e_{ss} = \lim_{t \to \infty} (r(t) - c(t))$$

Input signal	Error	Steady-state error
Unit-Step $r(t) = 1 \text{ for } t \geq 0$	$e(t) = e^{-t/T}$	$e_{ss} = 0$
Unit-Ramp $r(t) = t \text{ for } t \geq 0$	$e(t) = T\left(1 - e^{-t/T}\right)$	$e_{ss} = T$
Unit-Impulse $r(t) = \delta(t)$	$e(t) = \delta(t) - \frac{1}{T} e^{-t/T}$	$e_{ss} = 0$

Important Property of LTI Systems

Using unit-impulse, unit-step, unit-ramp the following outputs are obtained:

$$r_1(t)=\delta(t)=\begin{cases}\infty & t=0\\ 0 & \text{otherwise}\end{cases}$$

$$c_1(t)=\frac{1}{T}e^{-t/T} \quad \text{for} \quad t\geq 0$$

$$r_2(t)=1 \ \text{ for } t\geq 0$$

$$c_2(t)=1-e^{-t/T} \quad \text{for} \quad t\geq 0$$

$$r_3(t)=t \ \text{ for } t\geq 0$$

$$c_3(t)=t-T+Te^{-t/T} \quad \text{for} \quad t\geq 0$$

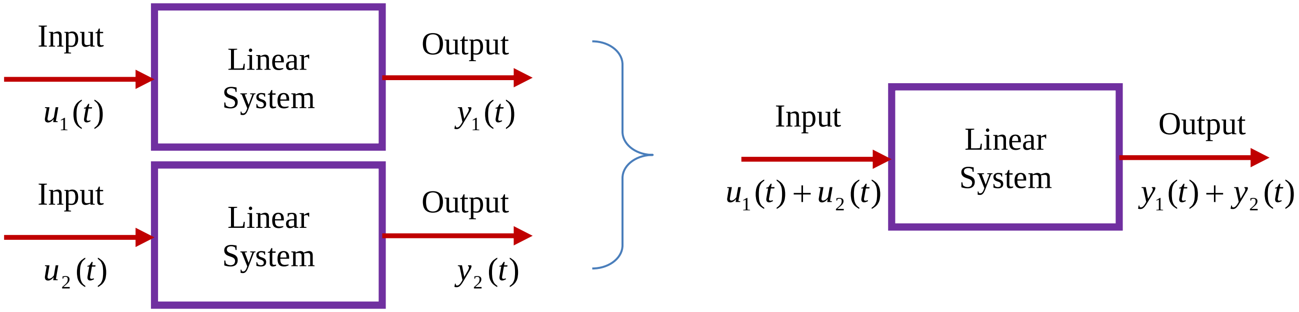
Since unit-impulse is the derivative of unit-step and unit-step is the derivative of unit-ramp, the corresponding outputs have the same relation in LTI systems:

$$r_1(t)=\frac{d}{dt}r_2(t) \quad \Rightarrow \quad c_1(t)=\frac{d}{dt}c_2(t)$$

$$r_2(t)=\frac{d}{dt}r_3(t) \quad \Rightarrow \quad c_2(t)=\frac{d}{dt}c_3(t)$$

Important Property of LTI Systems

- **Superposition**

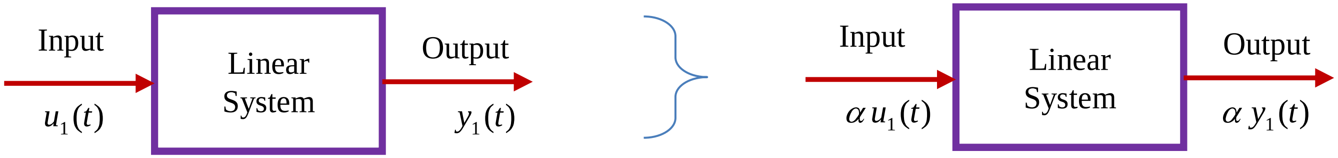


Input
$u_1(t)$
Linear System
Output
$y_1(t)$

Input
$u_2(t)$
Linear System
Output
$y_2(t)$

Input
$u_1(t)+u_2(t)$
Linear System
Output
$y_1(t)+y_2(t)$

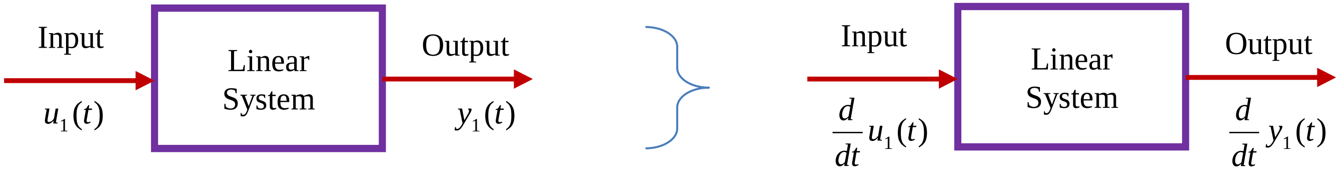
- **Homogeneity or Scaling**



Input
$u_1(t)$
Linear System
Output
$y_1(t)$

Input
$\alpha\, u_1(t)$
Linear System
Output
$\alpha\, y_1(t)$

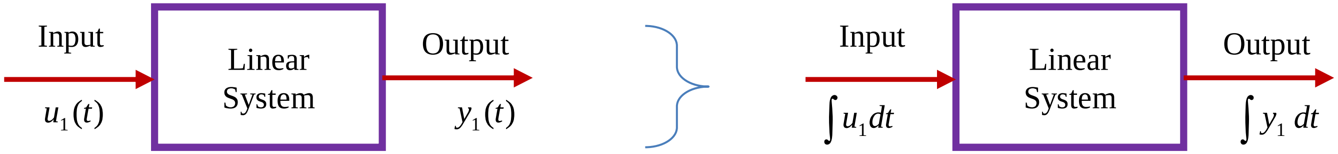
- **Derivative**



Input
$u_1(t)$
Linear System
Output
$y_1(t)$

Input
$\frac{d}{dt}u_1(t)$
Linear System
Output
$\frac{d}{dt}y_1(t)$

- **Integration**



Input
$u_1(t)$
Linear System
Output
$y_1(t)$

Input
$\int u_1 dt$
Linear System
Output
$\int y_1\, dt$

Second-Order System

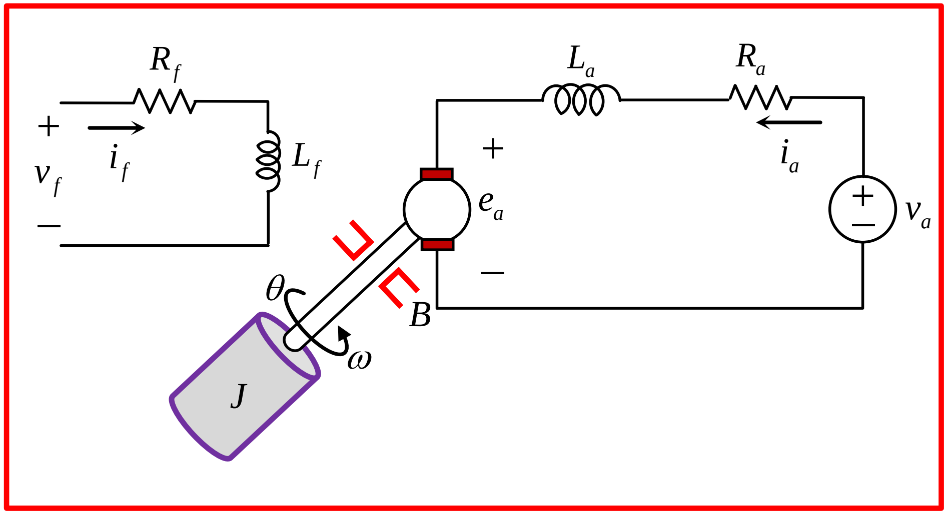
Consider the second-order system shown below. Physically, this system may represent DC servo drive system.

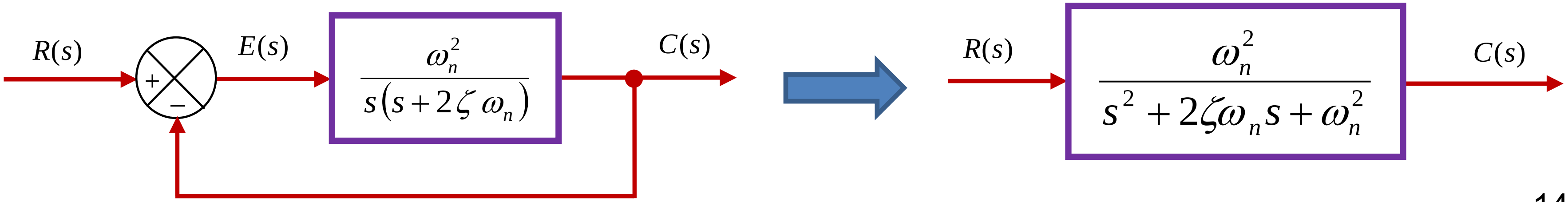
$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Standard Form

ω_n : Undamped natural frequency

ζ : Damping ratio



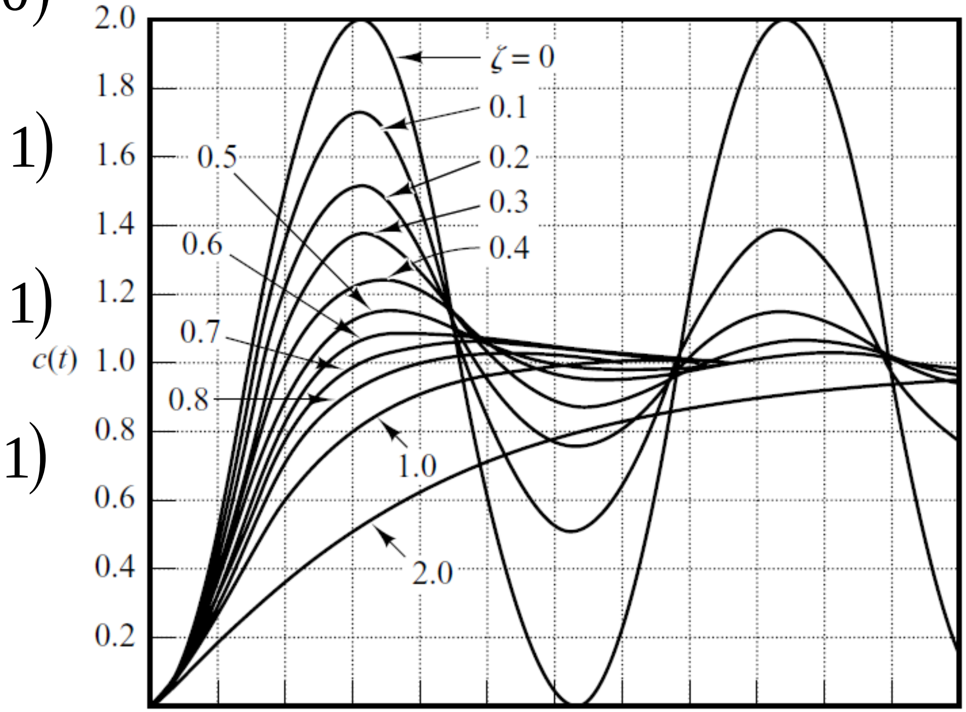


Second-Order System

The dynamic behaviour of the second-order system can be described in terms of two parameters ζ and ω_n.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

1. **Undamped case** $(\zeta = 0)$
2. **Under-damped case** $(0 < \zeta < 1)$
3. **Critically damped case** $(\zeta = 1)$
4. **Over-damped case** $(\zeta > 1)$



Second-Order System

1. Undamped case $(\zeta = 0)$

In this the oscillation continues indefinitely.

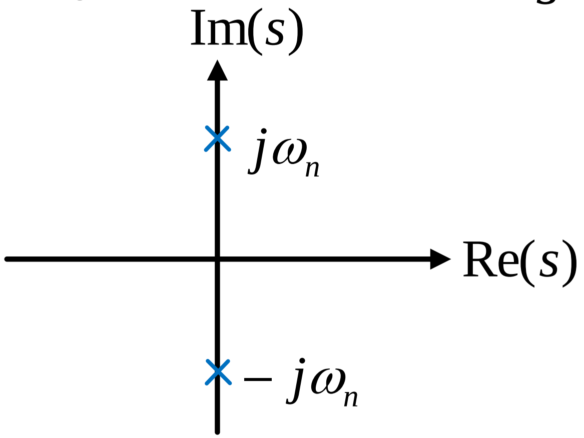
$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + \omega_n^2}$$

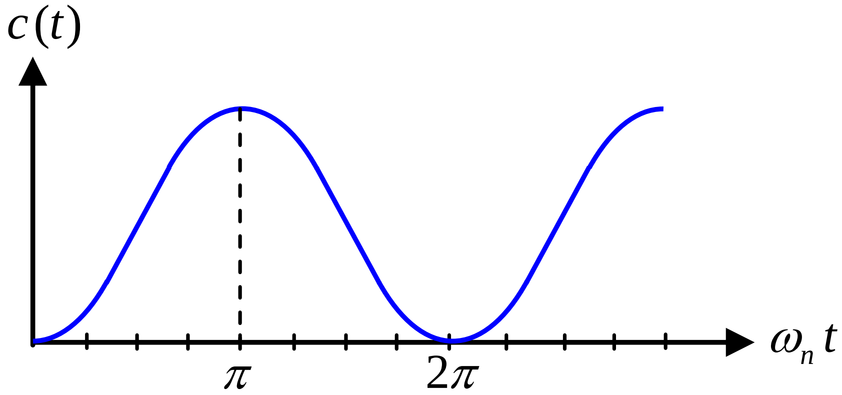
The roots of the denominator are on the imaginary axis

$$s_1 = j\omega_n \qquad\qquad s_2 = -j\omega_n$$

The unit-step response is obtained as

$$R(s) = \frac{1}{s} \quad \Rightarrow \quad C(s) = \frac{\omega_n^2}{s^2 + \omega_n^2}\frac{1}{s} \quad \Rightarrow \quad \boxed{c(t) = 1 - \cos(\omega_n t) \quad \text{for } \; t \geq 0}$$



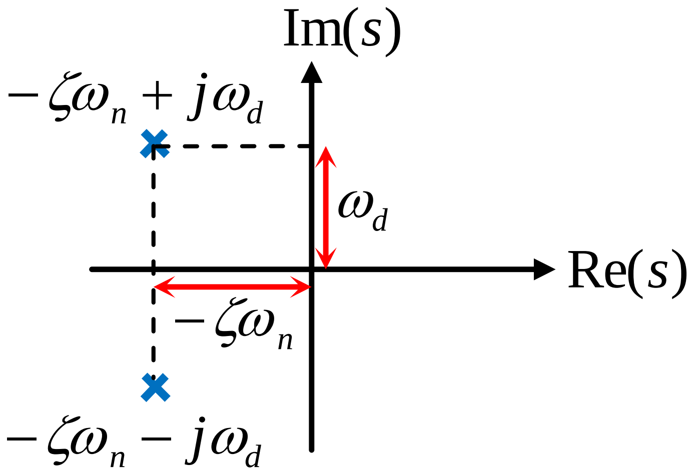


Second-Order System

2. Under-damped case $(0<\zeta<1)$

- In this case the closed-loop poles are complex conjugates and lie in the left-half s plane.

$$\frac{C(s)}{R(s)}=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$



- The transient response is oscillatory.

- The closed-loop transfer function can be written as

$$\frac{C(s)}{R(s)}=\frac{\omega_n^2}{\left(s+\zeta\omega_n+j\omega_d\right)\left(s+\zeta\omega_n-j\omega_d\right)}$$

$$s_1=-\zeta\omega_n+j\omega_d$$

$$s_1=-\zeta\omega_n-j\omega_d$$

where $\omega_d=\omega_n\sqrt{1-\zeta^2}$ is called the damped natural frequency.

Second-Order System

2. Under-damped case $(0<\zeta<1)$

The unit-step response of this case is as follows

$$R(s)=\frac{1}{s} \quad \Rightarrow \quad C(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\,\frac{1}{s}$$

$$\Rightarrow \quad C(s)=\frac{1}{s}-\frac{s+\zeta\omega_n}{\left(s+\zeta\omega_n\right)^2+\omega_d^2}-\frac{\zeta\omega_n}{\left(s+\zeta\omega_n\right)^2+\omega_d^2}$$

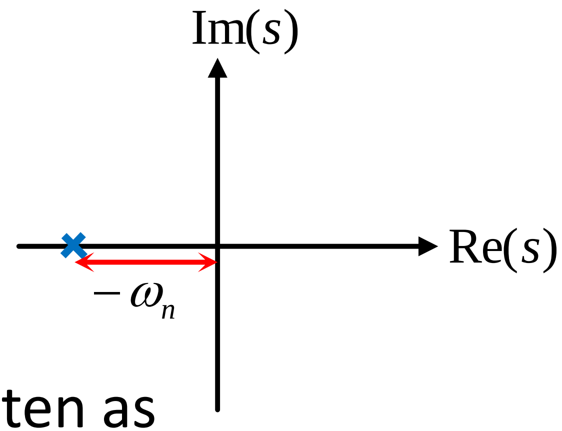
$$\Rightarrow \quad c(t)=1-e^{-\zeta\,\omega_n t}\left(\cos\omega_d t+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right) \quad \text{for} \;\; t\geq 0$$

$$\Rightarrow \quad \boxed{c(t)=1-\frac{e^{-\zeta\,\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t+\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right) \quad \text{for} \;\; t\geq 0}$$

Second-Order System

3. Critically-damped case $(\zeta = 1)$

- In this case the two poles are equal.



$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2}$$

- The closed-loop transfer function can be written as

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s + \omega_n)^2}$$

$$s_1 = s_2 = -\omega_n$$

- The unit-step response of this case is as follows

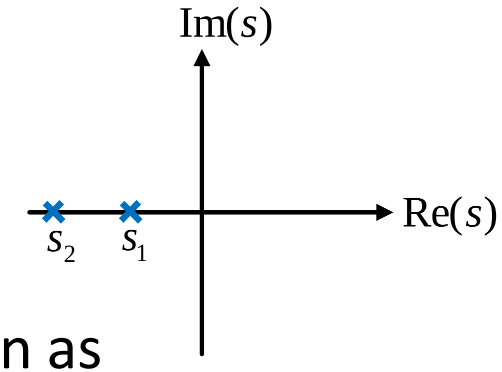
$$R(s) = \frac{1}{s} \Rightarrow C(s) = \frac{\omega_n^2}{(s+\omega_n)^2}\frac{1}{s} \Rightarrow \boxed{c(t) = 1 - e^{-\omega_n t}(1 + \omega_n t) \quad \text{for } t \geq 0}$$

Second-Order System

4. Over-damped case $(\zeta > 1)$

In this case the two poles are negative real and unequal.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



- The closed-loop transfer function can be written as

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{\left(s + \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}\right)\left(s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}\right)}$$

$$s_1 = -\left(\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n$$

$$s_2 = -\left(\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n$$

- The unit-step response of this case is as follows

$$R(s) = \frac{1}{s} \Rightarrow C(s) = \frac{\omega_n^2}{\left(s + \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}\right)\left(s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}\right)s}$$

Second-Order System

4. Over-damped case $(\zeta > 1)$

- The unit-step response of this case is as follows

$$R(s)=\frac{1}{s} \quad \Rightarrow \quad C(s)=\frac{\omega_n^2}{\left(s+\zeta\omega_n+\omega_n\sqrt{\zeta^2-1}\right)\left(s+\zeta\omega_n-\omega_n\sqrt{\zeta^2-1}\right)s}$$

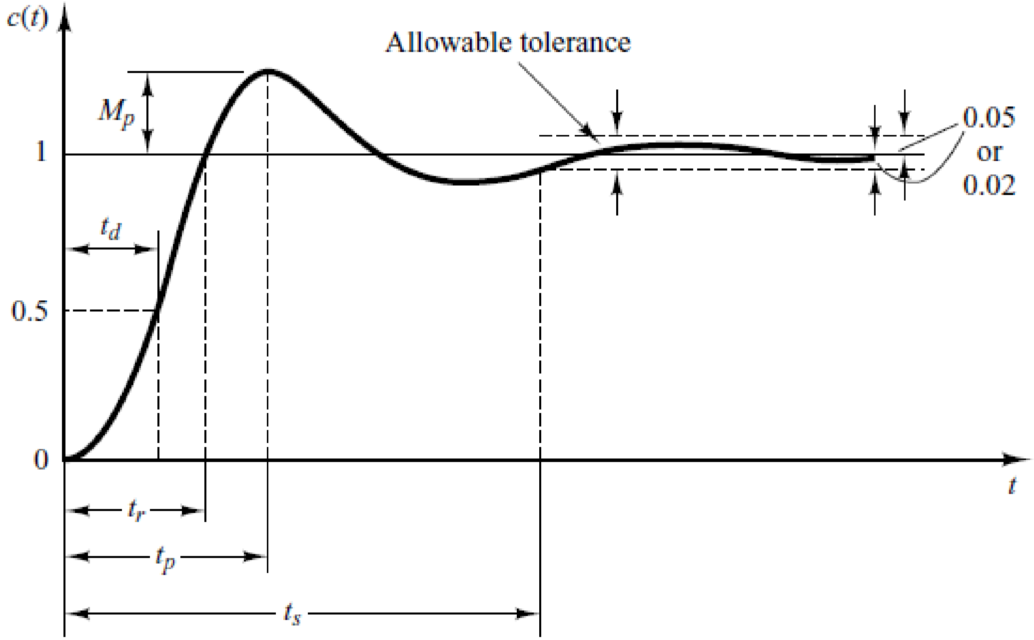
- Taking inverse Laplace transform yields

$$\boxed{c(t)=1-\frac{1}{2\sqrt{\zeta^2-1}}\left(\frac{e^{-\left(\zeta-\sqrt{\zeta^2-1}\right)\omega_n t}}{\zeta-\sqrt{\zeta^2-1}}-\frac{e^{-\left(\zeta+\sqrt{\zeta^2-1}\right)\omega_n t}}{\zeta+\sqrt{\zeta^2-1}}\right) \quad \text{for } t\ge 0}$$

Definitions of Transient-Response Specifications

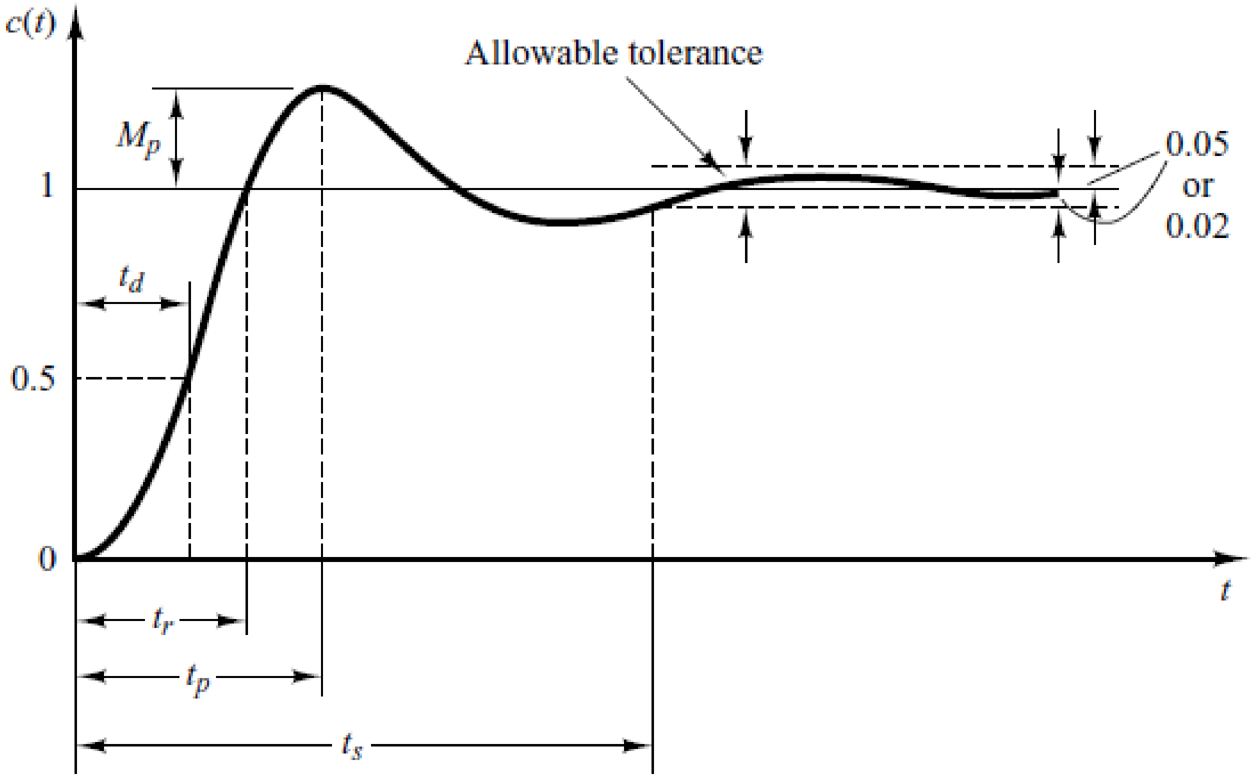
In specifying the transient-response characteristics of a control system to a unit-step input, it is common to specify the following:

1. **Delay time**, t_d
2. **Rise time**, t_r
3. **Peak time**, t_p
4. **Maximum overshoot**, M_p
5. **Settling time**, t_s



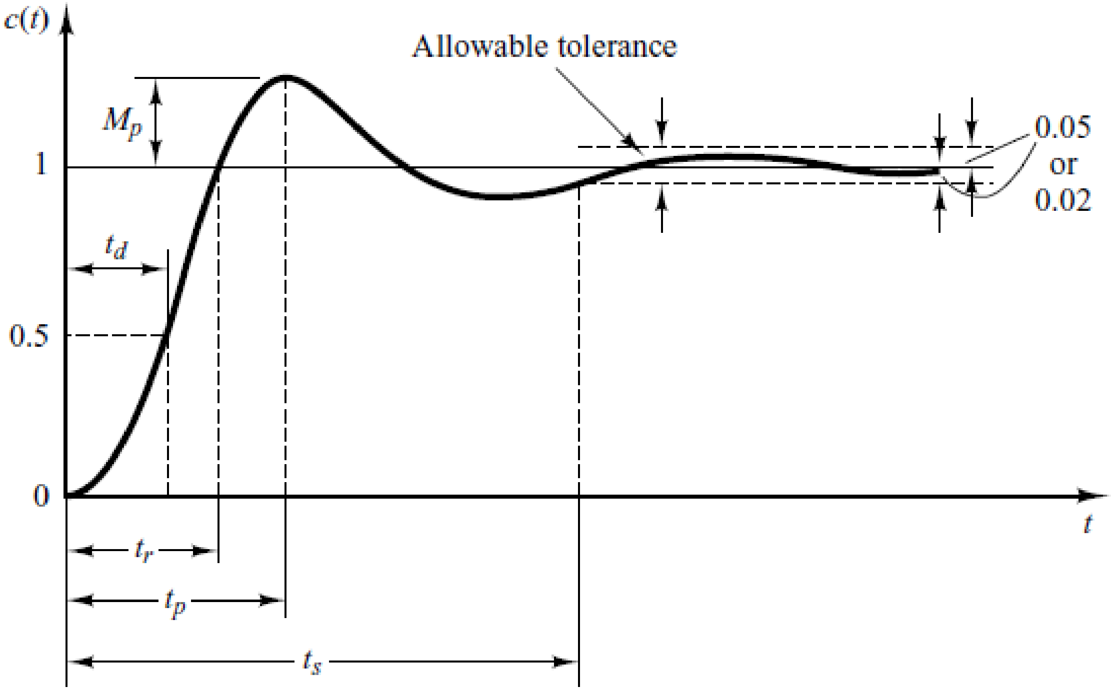
Definitions of Transient-Response Specifications

1. Delay time, t_d :The delay time is the time required for the response to reach half the final value the very first time.



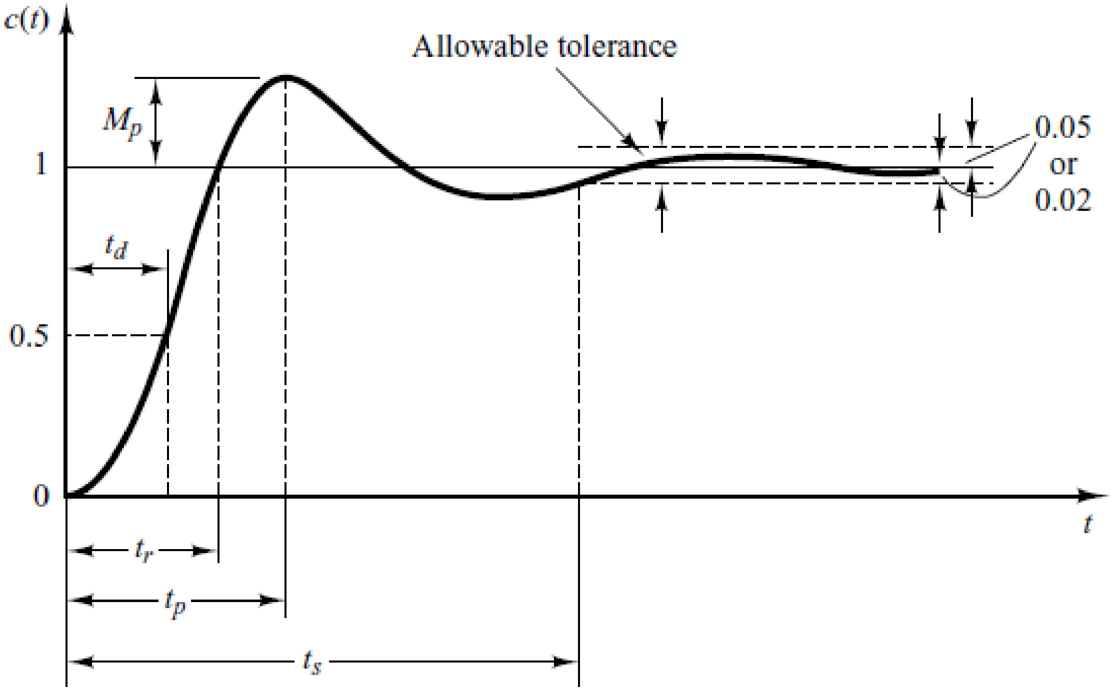
Definitions of Transient-Response Specifications

2. **Rise time, t_r**: The rise time is the time required for the response to rise from 10% to 90%, 5% to 95%, or 0% to 100% of its final value. For underdamped second order systems, the 0% to 100% rise time is normally used. For overdamped systems, the 10% to 90% rise time is commonly used.



Definitions of Transient-Response Specifications

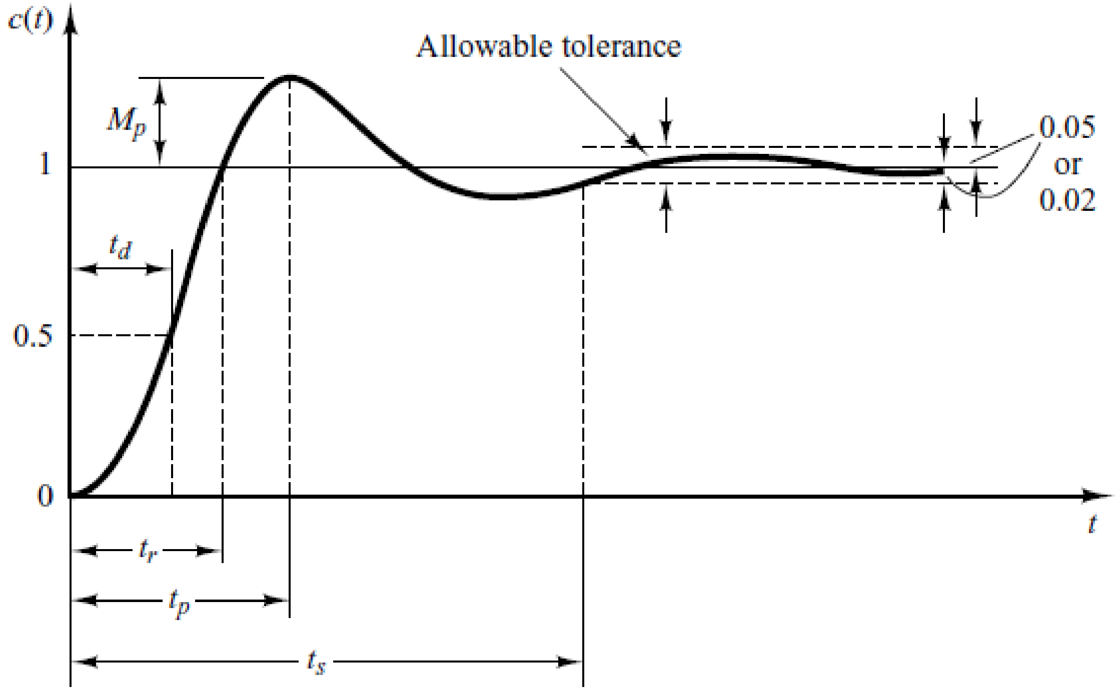
3. Peak time, t_p: The peak time is the time required for the response to reach the first peak of the overshoot.



Definitions of Transient-Response Specifications

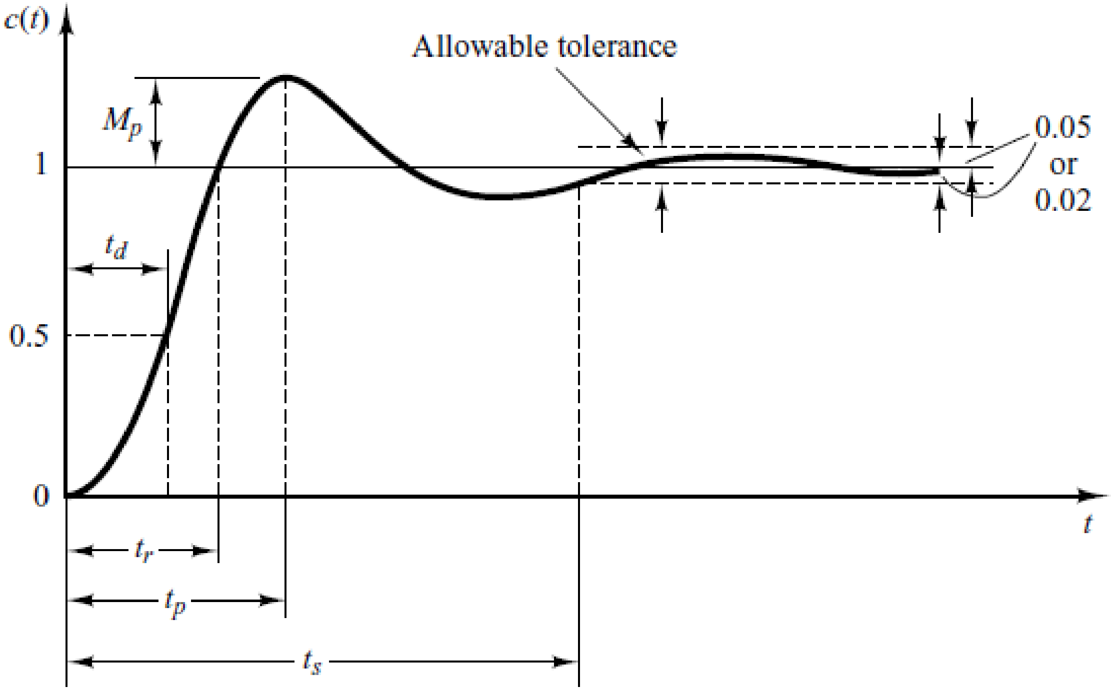
4. **Maximum overshoot, M_p**: The maximum overshoot is the maximum peak value of the response curve measured from unity. If the final steady-state value of the response differs from unity, then it is common to use the maximum percent overshoot. It is defined by

$$M_p\% = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100$$



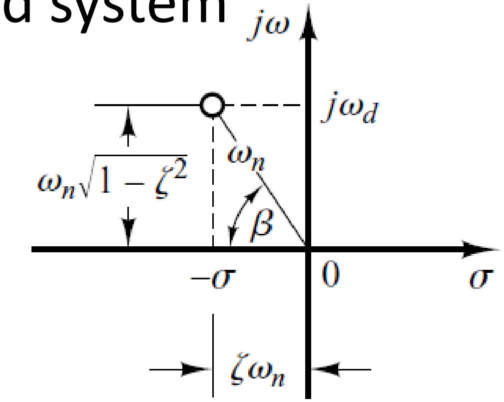
Definitions of Transient-Response Specifications

5. **Settling time, t_s**: The settling time is the time required for the response curve to reach and stay within a range about the final value of size specified by absolute percentage of the final value (usually 2% or 5%). The settling time is related to the largest time constant of the control system.



Second-Order Systems and Transient-Response Specifications

Rise time t_r: By letting $c(t_r)=1$ in an under-damped system



$$c(t_r)=1=1-\frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t_r+\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

$$\frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t_r+\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)=0$$

since $e^{-\zeta\omega_n t_r}\neq 0$ so

$$\sin\left(\omega_d t_r+\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)=0 \quad \Rightarrow \quad t_r=-\frac{1}{\omega_d}\tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

$$t_r=\frac{\pi-\cos^{-1}\zeta}{\omega_d}$$

where $\omega_d=\omega_n\sqrt{1-\zeta^2}$

Second-Order Systems and Transient-Response Specifications

Peak time t_p: Assume again the system is under-damped

$$\left.\frac{dc}{dt}\right|_{t=t_p} = 0 \quad \Longrightarrow \quad \frac{dc}{dt} = \zeta\,\omega_n \frac{e^{-\zeta\,\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d\, t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right) - \omega_d \frac{e^{-\zeta\,\omega_n t}}{\sqrt{1-\zeta^2}} \cos\left(\omega_d\, t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

$$\left.\frac{dc}{dt}\right|_{t=t_p} = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_d\, t_p\right) = 0 \quad \Longrightarrow \quad \sin\left(\omega_d\, t_p\right) = 0$$

$$\Longrightarrow \quad \omega_d\, t_p = 0,\ \pi,\ 2\pi,\ 3\pi \quad \xrightarrow{\text{1st peak}} \quad \boxed{t_p = \frac{\pi}{\omega_d}}$$

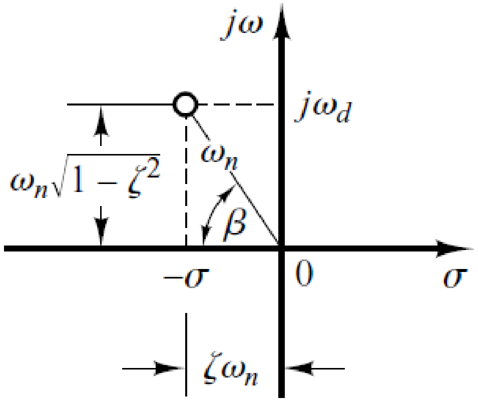
Second-Order Systems and Transient-Response Specifications

***Maximum overshoot M_p*:** Assume again the system is under-damped

$$M_p = c(t_p) - 1 \;\Rightarrow\; M_p = -\frac{e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}}\sin\left(\pi + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

$$M_p = -\frac{e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}}\left[\underbrace{\sin\pi}_{0}\cos\left(\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right) + \underbrace{\cos\pi}_{-\sqrt{1-\zeta^2}}\sin\left(\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right]$$

$$M_p = e^{-\zeta\omega_n(\pi/\omega_d)} \;\Rightarrow\; \boxed{M_p = e^{-\left(\zeta/\sqrt{1-\zeta^2}\right)\pi}}$$



Second-Order Systems and Transient-Response Specifications

Settling time t_s: Assume again the system is under-damped

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right) \quad \text{for } t \geq 0$$

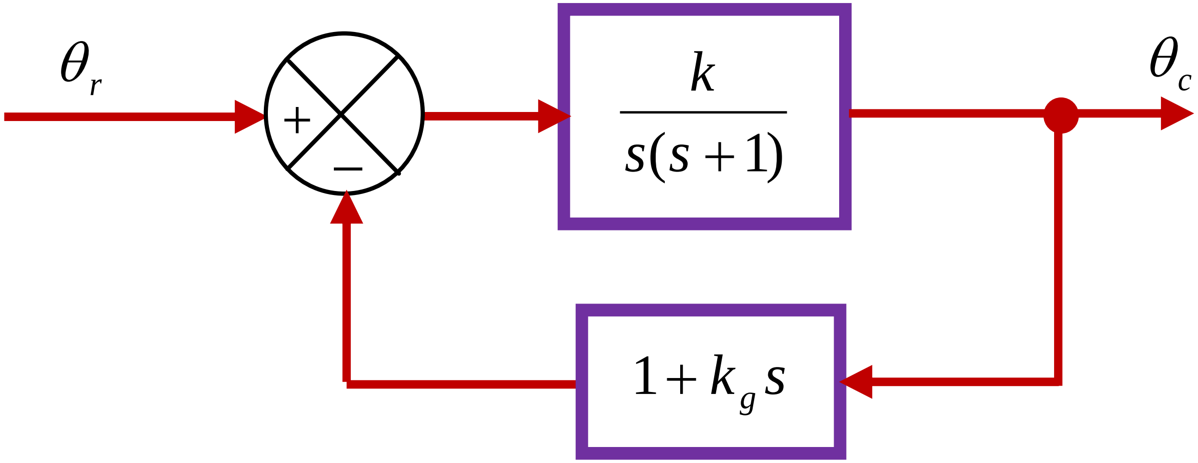
Consider the time constant is $T = \dfrac{1}{\zeta\omega_n}$

$$t_s = 4T = \frac{4}{\zeta\omega_n}$$ (2% criterion)

$$t_s = 3T = \frac{3}{\zeta\omega_n}$$ (5% criterion)

Second-Order Systems

Example: Consider a system with the following block diagram. Calculate k and k_g so that the maximum overshoot remains under 20% and the peak time happens before the first second for a unit-step input.



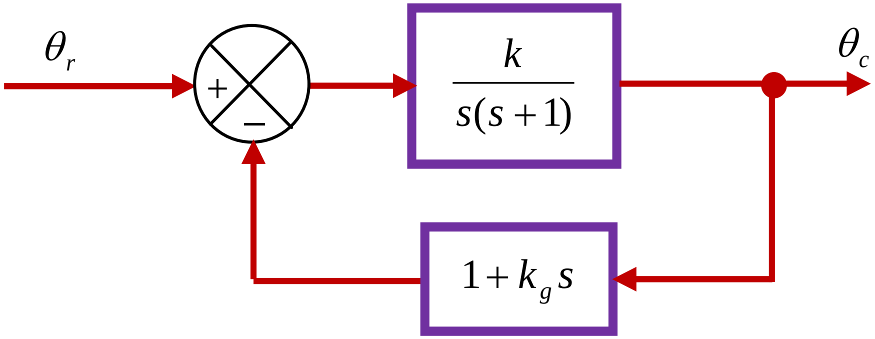
Second-Order Systems

Example: $k = ?$ $k_g = ?$ to have $M_p \le 20\%$ and $t_p \le 1$

$$\frac{\theta_c}{\theta_r} = \frac{\dfrac{k}{s(s+1)}}{1+\dfrac{k}{s(s+1)}(1+k_g s)} = \frac{k}{s^2+(1+kk_g)s+k}$$

$$\frac{\theta_c}{\theta_r} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$\begin{cases} \omega_n^2 = k \Longrightarrow \omega_n = \sqrt{k} \\ 2\zeta\omega_n = 1+kk_g \Longrightarrow \zeta = \dfrac{1+kk_g}{2\sqrt{k}} \end{cases}$$



Second-Order Systems

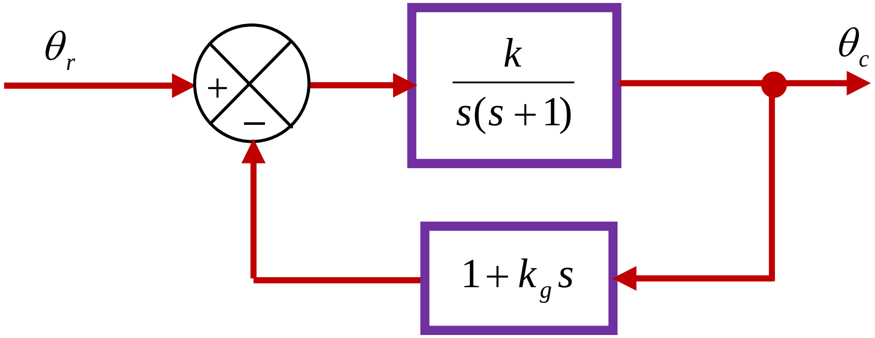
Example: $k = ?$ $k_g = ?$ to have $M_p \le 20\%$ and $t_p \le 1$

$$\omega_n = \sqrt{k} \qquad \zeta = \frac{1 + kk_g}{2\sqrt{k}}$$

$$M_p \le 20\% \Rightarrow M_p = e^{-\left(\zeta/\sqrt{1-\zeta^2}\right)\pi} \le 0.2 \Rightarrow \zeta \ge 0.456$$

$$t_p \le 1 \Rightarrow t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \le 1 \Rightarrow \omega_n \ge \frac{\pi}{\sqrt{1-\zeta^2}}$$

$$\Rightarrow \omega_n \ge 3.53 \quad \text{rad/s}$$

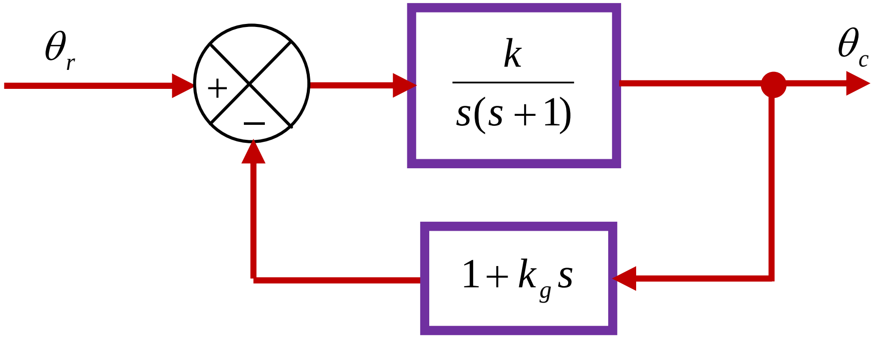


Second-Order Systems

Example: $k = ?$ $k_g = ?$ to have $M_p \leq 20\%$ and $t_p \leq 1$

$$\left.\begin{array}{l}\omega_n = \sqrt{k} \\ \omega_n \geq 3.53 \quad \text{rad/s}\end{array}\right\} \sqrt{k} \geq 3.53 \Rightarrow k \geq 12.5$$

$$\left.\begin{array}{l}\zeta = \dfrac{1+kk_g}{2\sqrt{k}} \\ \zeta \geq 0.456\end{array}\right\} \frac{1+kk_g}{2\sqrt{k}} \geq 0.456 \Rightarrow k_g \geq \frac{0.456 \times 2\sqrt{k} - 1}{k}$$



Second-Order System

The unit-impulse response of the second-order system can be easily obtained by the inverse Laplace transform of

$$C(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- For $0 \le \zeta < 1$ $\qquad \omega_d = \omega_n\sqrt{1-\zeta^2}$

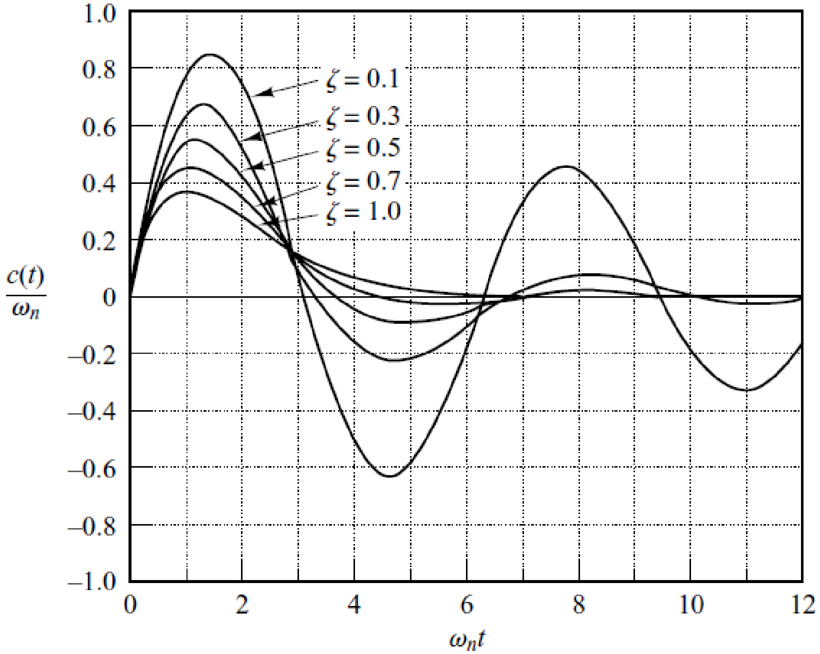
$$c(t) = \frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t) \quad \text{for} \;\; t \ge 0$$

- For $\zeta = 1$

$$c(t) = \omega_n^2 t\, e^{-\omega_n t} \qquad \text{for} \;\; t \ge 0$$

- For $\zeta > 1$

$$c(t) = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} \left(e^{-\left(\zeta - \sqrt{\zeta^2-1}\right)\omega_n t} - e^{-\left(\zeta + \sqrt{\zeta^2-1}\right)\omega_n t} \right) \qquad \text{for} \;\; t \ge 0$$



Second-Order Systems and Transient-Response Specifications

The maximum overshoot for the unit-impulse response of the under-damped system occurs at

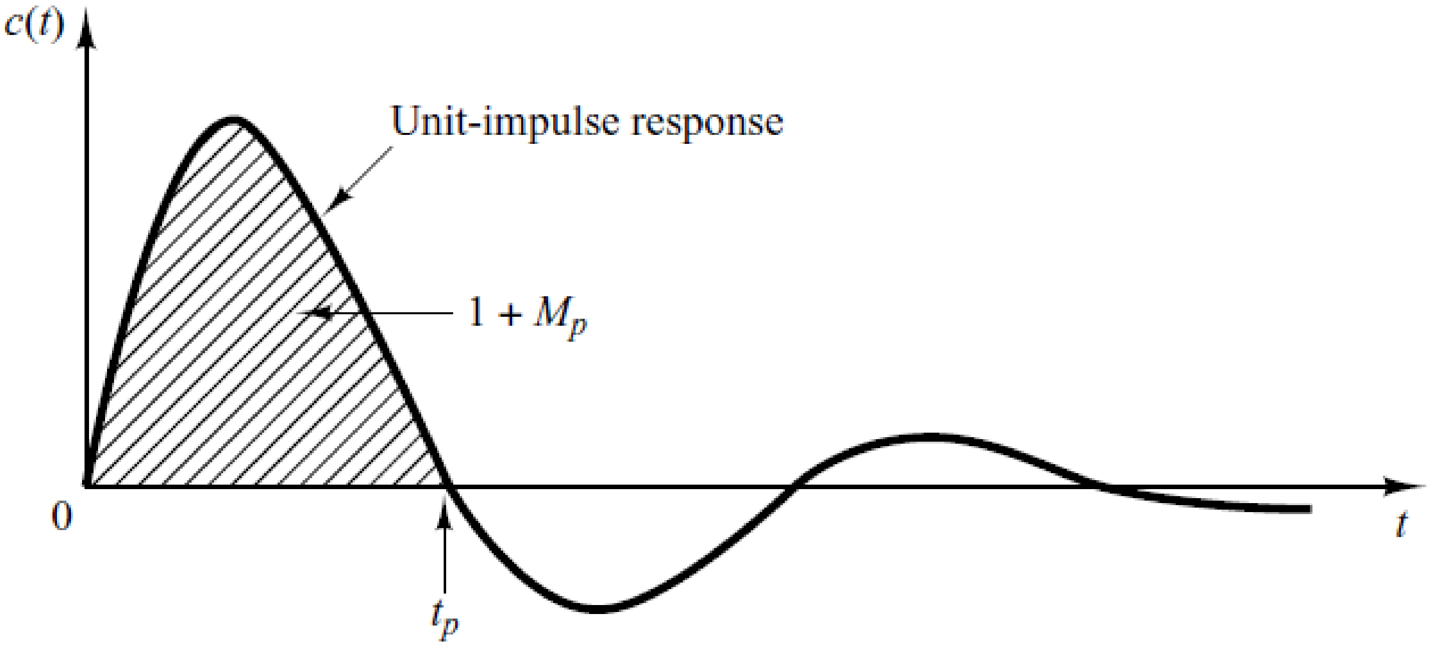
$$t_{peak} = \frac{\tan^{-1}\dfrac{\sqrt{1-\zeta^2}}{\zeta}}{\omega_n\sqrt{1-\zeta^2}} \qquad \text{where} \;\; 0<\zeta<1$$

The maximum overshoot is

$$c(t)_{\max} = \omega_n \exp\left(-\frac{\zeta}{\sqrt{1-\zeta^2}}\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right) \qquad \text{where} \;\; 0<\zeta<1$$

Second-Order Systems

The peak time (t_p) and maximum overshoot (M_p) of a unit-step response can be obtained from the unit-impulse response



Higher-Order Systems

- The closed-loop transfer function of higher-order systems can be expressed using the following general form:

$$\frac{C(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} \qquad (m \le n)$$

- Factorizing the numerator and denominator yields

$$\frac{C(s)}{R(s)} = \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} \qquad (m \le n)$$

- The roots of the numerator are the zeros (z_j where j=1,2,…,m) of the system and the roots of the denominator are the poles (p_i where i=1,2,…,n) of the system.

Higher-Order Systems

- Assuming all **poles** are **real and distinct**, for a unit-step input the output is

$$C(s)=\frac{a}{s}+\sum_{i=1}^{n}\frac{a_i}{s+p_i}$$

where a_i is the residue of the pole at $s = -p_i$

- If the system involves multiple poles, then $C(s)$ will have multiple-pole terms.

Higher-Order Systems

- If all closed-loop poles lie in the **left-half** *s* plane, the relative **magnitudes** of the residues determine the relative **importance** of the components in the expanded form of $C(s)$.

- If there is a closed-loop **zero close** to a closed-loop **pole**, then the residue at this pole is small and the coefficient of the transient-response term corresponding to this pole becomes small.

- A pair of closely located poles and zeros will effectively **cancel** each other.

If a pole is close to a zero, the effect of that pole is low.

Higher-Order Systems

- If a pole is located very **far** from the origin, the residue at this pole may be **small**.

- The transients corresponding to such a **remote** pole are **small** and last a short time.

- Terms in the expanded form of $C(s)$ having very **small residues contribute little** to the transient response, and these terms may be neglected.

- If this is done, the higher-order system may be approximated by a lower-order one.

Far poles from the origin have low effects.

Higher-Order Systems

- Consider the case where the poles of $C(s)$ consist of **distinct real poles** and **pairs of complex-conjugate poles**. Therefore we have

$$C(s)=\frac{a}{s}+\sum_{j=1}^{q}\frac{a_j}{s+p_j}+\sum_{k=1}^{r}\frac{b_k\left(s+\zeta_k\omega_k\right)+c_k\omega_k\sqrt{1-\zeta_k^2}}{s^2+2\zeta_k\omega_k s+\omega_k^2}\qquad (q+2r=n)$$

- The unit-step response $c(t)$ is then

$$c(t)=a+\sum_{j=1}^{q}a_je^{-p_jt}+\sum_{k=1}^{r}b_ke^{-\zeta_k\omega_kt}\cos\left(\omega_k\sqrt{1-\zeta_k^2}\,t\right)$$
$$+\sum_{k=1}^{r}c_ke^{-\zeta_k\omega_kt}\sin\left(\omega_k\sqrt{1-\zeta_k^2}\,t\right)$$

- If all poles have negative real part, the system is stable and $c(\infty)=a$

Dominant Closed-Loop Poles

- If the ratios of the real parts of the poles exceed 5 and there are no zeros nearby, the poles nearest the $j\omega$ axis will dominate the transient response behaviour.

- **Example**: In the following system the **dominant pole is 0.5**, because there is no zero nearby (near 0.5) and the ratio of poles is 6 which exceeds 5.

$$\frac{C(s)}{R(s)} = \frac{7(s+5)}{(s+3)(s+0.5)}$$

- **Example**: In the following system a zero is near the pole 0.5 therefore the dominant pole is not 0.5 but it is 3.

$$\frac{C(s)}{R(s)} = \frac{s(s+0.48)}{(s+3)(s+0.5)}$$

Stability Analysis in the Complex Plane

- If **all** closed-loop poles lie in the **left-half *s* plane**, the system is **stable**.

$$\frac{C(s)}{R(s)}=\frac{7(s+5)}{(s+3)(s+0.5)}$$

- If **any** of the closed-loop poles lie in the **right-half *s* plane**, the system is **unstable**.

$$\frac{C(s)}{R(s)}=\frac{7(s+5)}{(s+3)(s-0.5)}$$

Routh's Stability Criterion

Routh's stability criterion tells us whether or not there are unstable roots in a polynomial equation without actually solving for them. The procedure is as follows:

1. Consider the following close-loop system

$$\frac{C(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

2. Write the characteristics equation

$$a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0$$

Where the coefficients are real quantities. We assume that a_n is not zero; i.e. any zero root has been removed.

Routh’s Stability Criterion

3. If any of the coefficients are **zero or negative** in the presence of at least one positive coefficient, a root or roots exist that are imaginary or that have positive real parts. Therefore, in such a case, the system is not stable.

4. If all coefficients are positive, arrange the coefficients of the polynomial in rows and columns according to the following pattern:

The number of rows is $n+1$.

$$\begin{array}{c|ccccc}
s^n & a_0 & a_2 & a_4 & a_6 & \cdots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdots \\
\hline
s^{n-2} & & & & & \\
s^{n-3} & & & & & \\
\vdots & & & & & \\
s^0 & & & & &
\end{array}$$

$$\boxed{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0}$$

Routh's Stability Criterion

5. The coefficients to be calculated are listed in the table

$$\boxed{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0}$$

$$\begin{array}{l|lllll}
s^n & a_0 & a_2 & a_4 & a_6 & \cdots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdots \\
\hline
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
s^0 & g_1 & & & &
\end{array}$$

where

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1} \qquad b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1} \qquad b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1} \qquad \cdots$$

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1} \qquad c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1} \qquad c_3 = \frac{b_1 a_7 - a_1 b_4}{b_1} \qquad \cdots$$

Routh's Stability Criterion

- To simplify the calculation an entire row may be **divided or multiplied** by a **positive number**, e.g. $k > 0$.

- Routh's stability criterion states that the **number of roots with positive real parts** is equal to the **number of changes in sign** of the coefficients of the **first column** of the array.

- The necessary and sufficient **condition** that **all roots lie in the left-half s plane** is that **all terms in the first column of the array have positive signs**.

$$
\begin{array}{l|lllll}
s^n & a_0 & a_2 & a_4 & a_6 & \cdots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdots \\
\hline
s^{n-2} & kb_1 & kb_2 & kb_3 & kb_4 & \cdots \\
\cline{2-5}
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
s^0 & g_1 & & & &
\end{array}
$$

Routh’s Stability Criterion

Example: Apply Routh’s stability criterion to the following polynomial:

$$s^4+2s^3+3s^2+4s+5=0$$

1. Form the table and simplify (second row is divided by 2)

s^4	1	3	5
s^3	2	4	0
s^2			
s^1			
s^0			

s^4	1	3	5
s^3	1	2	0
s^2			
s^1			
s^0			

$$a_0s^n+a_1s^{n-1}+\cdots+a_{n-1}s+a_n=0$$

s^n	a_0	a_2	a_4	a_6	$\cdots$
s^{n-1}	a_1	a_3	a_5	a_7	$\cdots$
s^{n-2}					
s^{n-3}					
$\vdots$					
s^0					

Routh's Stability Criterion

Solution:

2. Calculate the remaining coefficients

$$\begin{array}{c|ccc} s^4 & 1 & 3 & 5 \\ s^3 & 1 & 2 & 0 \\ \hline s^2 & 1 & 5 & \\ s^1 & -3 & & \\ s^0 & 5 & & \end{array}$$

$$b_1 = \frac{1\times 3 - 1\times 2}{1} = 1$$

$$b_2 = \frac{1\times 5 - 1\times 0}{1} = 5$$

$$c_1 = \frac{1\times 2 - 1\times 5}{1} = -3$$

$$d_1 = \frac{-3\times 5 - 1\times 0}{-3} = 5$$

$$\begin{array}{c|ccccc} s^n & a_0 & a_2 & a_4 & a_6 & \cdots \\ s^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdots \\ \hline s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\ s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ s^0 & g_1 & & & & \end{array}$$

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

$$d_1 = \frac{c_1 b_2 - b_1 c_2}{c_1}$$

Routh's Stability Criterion

Solution:

- The first column numbers have **changed their signs twice**; therefore there are **two roots with positive real parts**.
- The system is therefore **unstable**.

$$
\begin{array}{l|ccc}
s^4 & 1 & 3 & 5 \\
s^3 & 1 & 2 & 0 \\
\hline
s^2 & 1 & 5 & \\
s^1 & -3 & & \\
s^0 & 5 & &
\end{array}
$$

Routh's Stability Criterion

Special Case 1: If a first-column term in any row is **zero**, but the remaining terms are not zero or there is no remaining term, then the zero term is replaced by a very small positive number ε and the rest of the array is evaluated.

Example: $s^3+2s^2+s+2=0$ →

$$\begin{array}{c|cc} s^3 & 1 & 1 \\ s^2 & 2 & 2 \\ \hline s^1 & 0\approx\varepsilon & \\ s^0 & 2 & \end{array}$$

If the sign of the coefficient above the zero (ε) is the same as that below it, it indicates that there are a **pair of imaginary roots**. Actually, This example has two roots at $s=\pm j$.

Routh's Stability Criterion

If, however, the sign of the coefficient above the zero (ε) is opposite that below it, it indicates that there is one sign change.

Example: $s^3 - 3s + 2 = (s-1)^2(s+2) = 0$

$$\begin{array}{c|cc} s^3 & 1 & -3 \\ s^2 & 0 \approx \varepsilon & 2 \\ \hline s^1 & -3 - \frac{2}{\varepsilon} & \\ s^0 & 2 & \end{array}$$

There are two sign changes of the coefficients in the first column. So there are two roots in the right-half s plane. This agrees with the correct result indicated by the factored form of the polynomial equation.

Routh's Stability Criterion

Special Case 2: If all the coefficients in any **derived row are zero**, it indicates that there are roots of equal magnitude lying radially opposite in the *s* plane (that is, two real roots with equal magnitudes and opposite signs and/or two conjugate imaginary roots).

In such a case, the evaluation of the rest of the array can be continued by forming an **auxiliary polynomial with the coefficients of the last row** and by using the coefficients of the derivative of this polynomial in the next row.

Such roots with equal magnitudes and lying radially opposite in the *s* plane can be found by solving the auxiliary polynomial, which is always even.

For a 2n-degree auxiliary polynomial, there are n pairs of equal and opposite roots.

Routh’s Stability Criterion

Example: $s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0$

$$\begin{array}{c|ccc} s^5 & 1 & 24 & -25 \\ s^4 & 2 & 48 & -50 \\ \hline s^3 & 0 & 0 & \end{array}$$

← Auxiliary polynomial $P(s)$

- The terms in the s^3 row are all zero. (Such a case occurs only in an odd-numbered row.)
- The auxiliary polynomial is formed from the coefficient of the s^4 row: $P(s) = 2s^4 + 48s^2 - 50$
- which indicates that there are two pairs of roots of equal magnitude and opposite sign (that is, two real roots with the same magnitude but opposite signs or two complex conjugate roots on the imaginary axis).

Routh’s Stability Criterion

Solution: $s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0$

- The derivative of $P(s)$ with respect to s is $\dfrac{dP(s)}{ds} = 8s^3 + 96s$
- The terms in the s^3 row are replaced by the coefficients of the last equation, i.e. 8 and 96.

s^5	1	24	-25
s^4	2	48	-50
s^3	8	96	
s^2	24	-50	
s^1	113	0	
s^0	-50		

← Coefficients of $dP(s)/dt$

Routh's Stability Criterion

Solution: $s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0$

- We see that there is one change in sign in the first column of the new array.
- Thus, the original equation has one root with a positive real part.

$$
\begin{array}{c|ccc}
s^5 & 1 & 24 & -25 \\
s^4 & 2 & 48 & -50 \\
\hline
s^3 & 8 & 96 & \\
s^2 & 24 & -50 & \\
s^1 & 113 & 0 & \\
s^0 & -50 & &
\end{array}
$$

Application of Routh's Stability Criterion to Control-System Analysis

Example: Consider the following system. Determine the range of K for stability.

$$\frac{C(s)}{R(s)} = \frac{K}{s^4 + 3s^3 + 3s^2 + 2s + K}$$

- The characteristics equations is $s^4 + 3s^3 + 3s^2 + 2s + K = 0$
- The array of coefficients becomes

$$\begin{array}{l|ccc}
s^4 & 1 & 3 & K \\
s^3 & 3 & 2 & 0 \\
\hline
s^2 & \frac{7}{3} & K & \\
s^1 & 2 - \frac{9}{7}K & & \\
s^0 & K & &
\end{array}$$

The numbers on the first column should all be positive:

$$\left.\begin{array}{l} 2 - \frac{9}{7}K > 0 \\ \\ K > 0 \end{array}\right\} \quad \frac{14}{9} > K > 0$$

Some Definitions

1. **Steady-State Response**

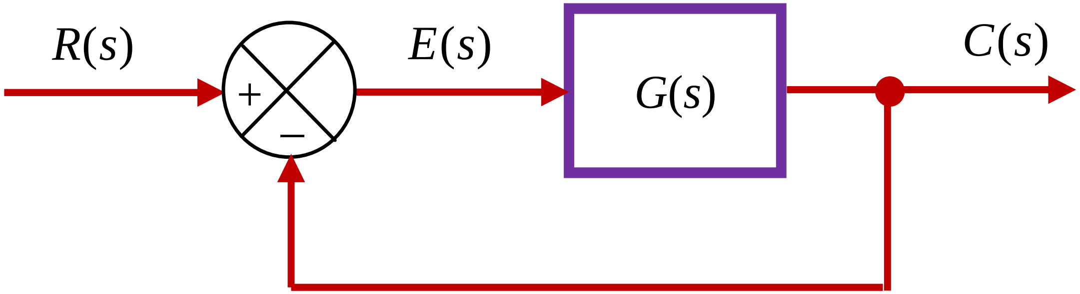
$$c_{ss}(t) = c(\infty) = \lim_{t\to\infty} c(t)$$

2. **Steady-State Error**

$$e_{ss}(t) = \lim_{t\to\infty} e(t)$$

or

$$e_{ss}(t) = \lim_{s\to 0} sE(s)$$

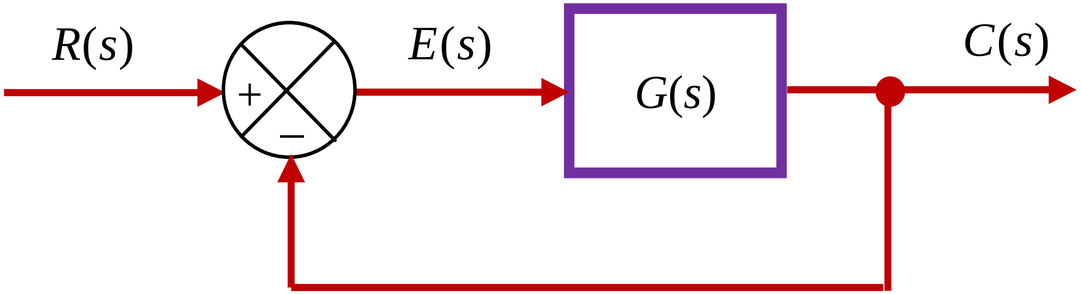


Some Definitions

3. **System Type:** Consider the unity-feedback control system with the following **open-loop transfer function** $G(s)$:

$$G(s)=\frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{s^N(s+p_1)(s+p_2)\cdots(s+p_q)}$$

- A system is called type 0, type 1, type 2, …, if N=0, N=1, N=2, …, respectively.



- In non-unity feedback control system, the system type is obtained from the open-loop transfer function $G(s)H(s)$.

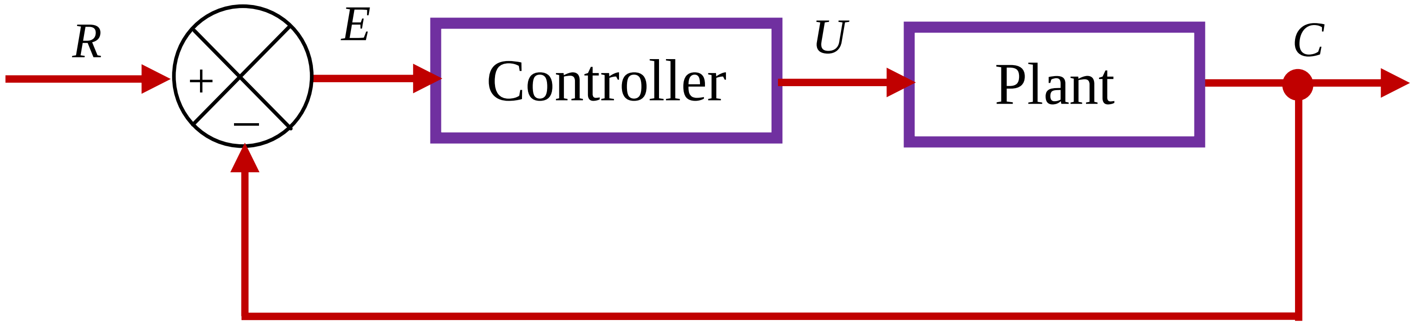
Different Types of Controllers

1. **Proportional controller (P)**

2. **Proportional-Integral controller (PI)**

3. **Proportional-Derivative controller (PD)**

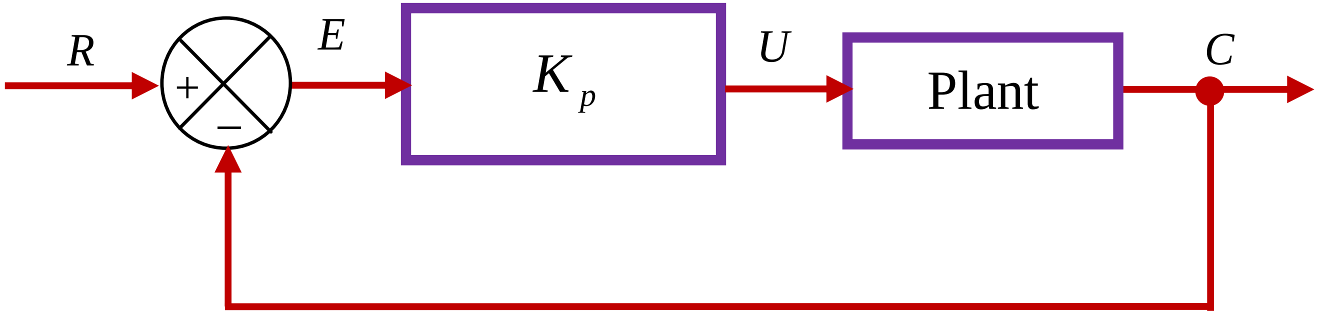
4. **Proportional-Integral-Derivative controller (PID)**



Different Types of Controllers

1. Proportional controller (P)

$u(t) = K_p e(t)$ ⇒ $U(s) = K_p E(s)$



Different Types of Controllers

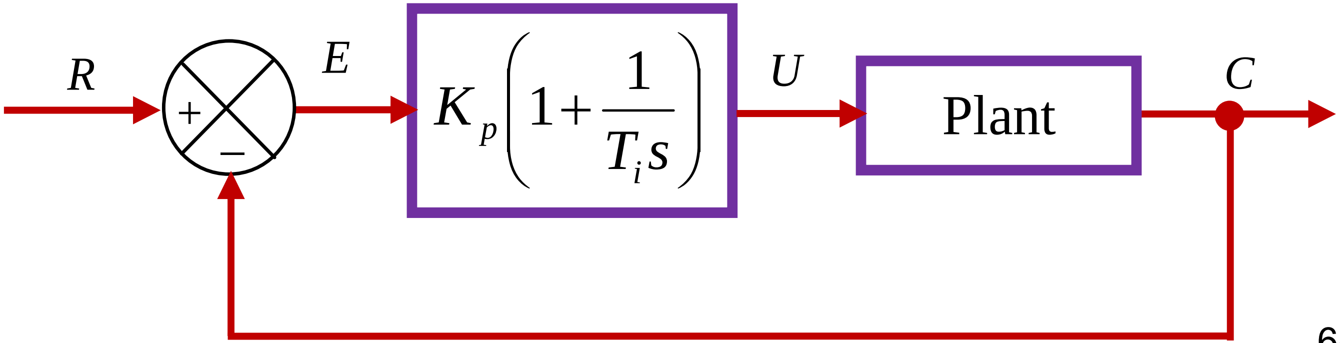
2. Proportional-Integral controller (PI)

$$u(t) = K_p e(t) + K_i \int_0^t e(\tau) d\tau \quad \Rightarrow \quad U(s) = \left(K_p + \frac{K_i}{s} \right) E(s)$$

Or

$$u(t) = K_p \left(e(t) + \frac{1}{T_i} \int_0^t e(\tau) d\tau \right) \quad \Rightarrow \quad U(s) = K_p \left(1 + \frac{1}{T_i s} \right) E(s)$$

$$K_i = \frac{K_p}{T_i}$$



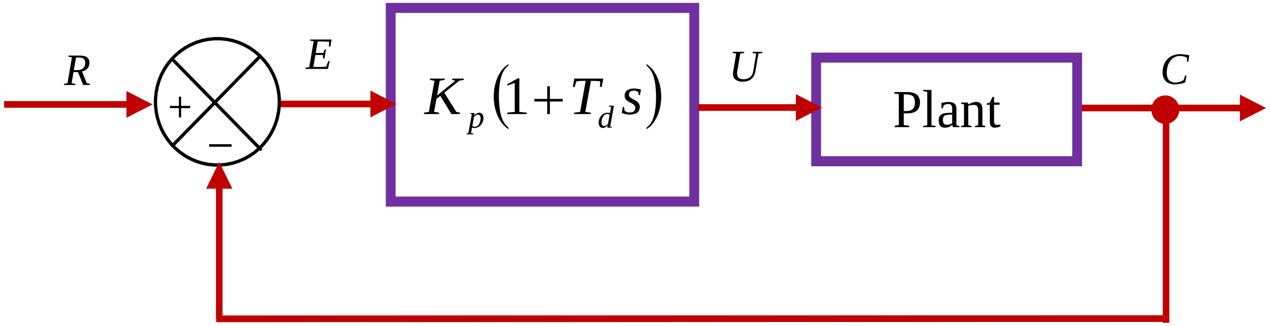
Different Types of Controllers

3. Proportional-Derivative controller (PD)

$$u(t) = K_p e(t) + K_d \frac{de(t)}{dt} \quad \Rightarrow \quad U(s) = \left(K_p + K_d\, s\right)E(s)$$

Or

$$u(t) = K_p \left(e(t) + T_d \frac{de(t)}{dt} \right) \quad \Rightarrow \quad U(s) = K_p \left(1 + T_d\, s\right)E(s)$$



$$K_d = K_p T_d$$

Different Types of Controllers

4. Proportional-Integral-Derivative controller (PID)

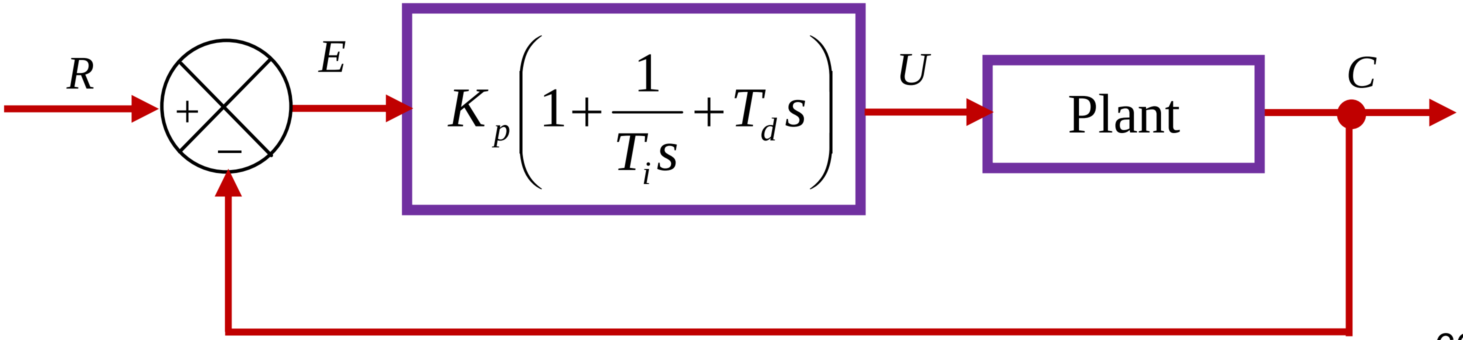
$$u(t) = K_p e(t) + K_i \int_0^t e(\tau) d\tau + K_d \frac{de(t)}{dt} \quad \Rightarrow \quad U(s) = \left(K_p + \frac{K_i}{s} + K_d\, s \right) E(s)$$

Or

$$u(t) = K_p \left(e(t) + \frac{1}{T_i} \int_0^t e(\tau) d\tau + T_d \frac{de(t)}{dt} \right) \quad \Rightarrow \quad U(s) = K_p \left(1 + \frac{1}{T_i s} + T_d\, s \right) E(s)$$

$$K_i = \frac{K_p}{T_i}$$

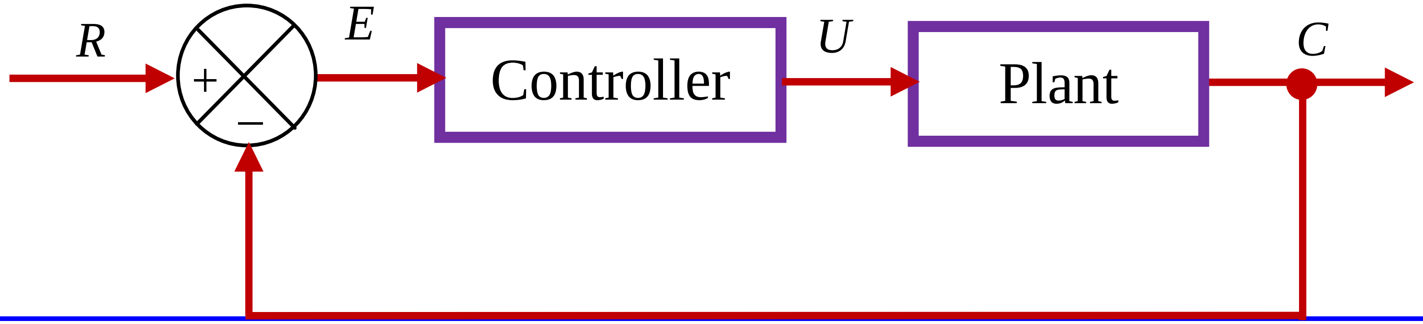
$$K_d = K_p T_d$$



Different Types of Controllers

Integral control action:

- In the proportional control of a plant whose transfer function does not possess an integrator 1/s, there is a steady-state error, or offset, in the response to a step input. Such an **offset can be eliminated** if the **integral** control action is included in the controller.
- Note that **integral** control action, while removing steady-state error, may **lead to oscillatory response** of slowly decreasing amplitude or even increasing amplitude, both of which are usually undesirable.



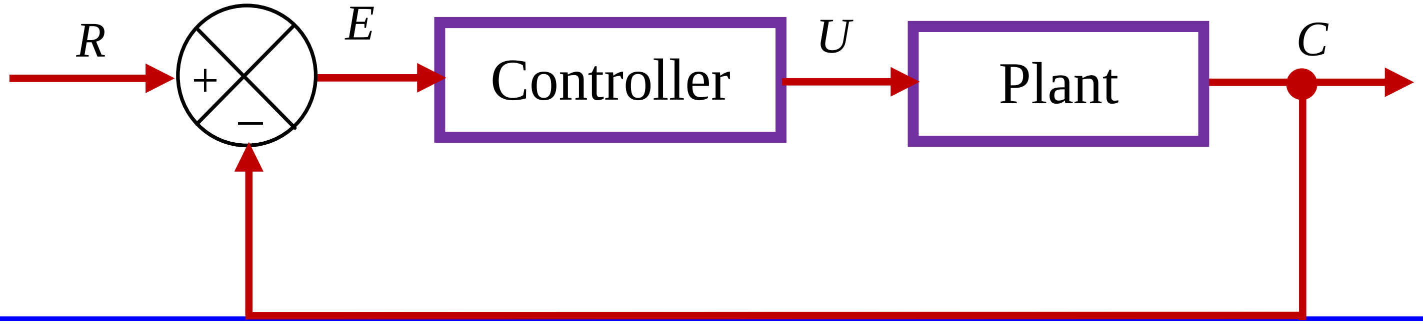
Different Types of Controllers

Derivative control action:

Derivative control action, when added to a proportional controller, provides a means of obtaining a controller with high **sensitivity**.

An advantage of using derivative control action is that it **responds** to the **rate of change of the actuating error** and can produce a significant correction before the magnitude of the actuating error becomes too large.

Derivative control thus **anticipates the actuating error**, initiates an early corrective action, and tends to increase the stability of the system.



Steady-State Errors in Unity-Feedback Control Systems

$$E(s)=R(s)-C(s) \Rightarrow E(s)=R(s)-R(s)\frac{G(s)}{1+G(s)} \Rightarrow E(s)=\frac{R(s)}{1+G(s)}$$

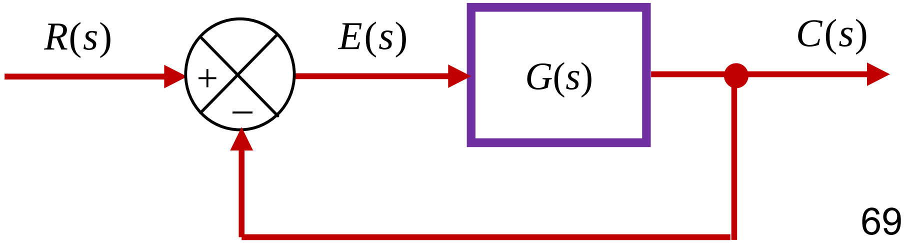
$$e_{ss}=\lim_{s\to 0} s\frac{R(s)}{1+G(s)}$$

Unit step input: $R(s)=\frac{1}{s}$

$$e_{ss}=\lim_{s\to 0} s\frac{1}{1+G(s)}\frac{1}{s} \Rightarrow e_{ss}=\frac{1}{1+G(0)} \Rightarrow e_{ss}=\frac{1}{1+k_p}$$

where k_p is the static position error constant

$$k_p=\lim_{s\to 0} G(s)=G(0)$$



Steady-State Errors in Unity-Feedback Control Systems

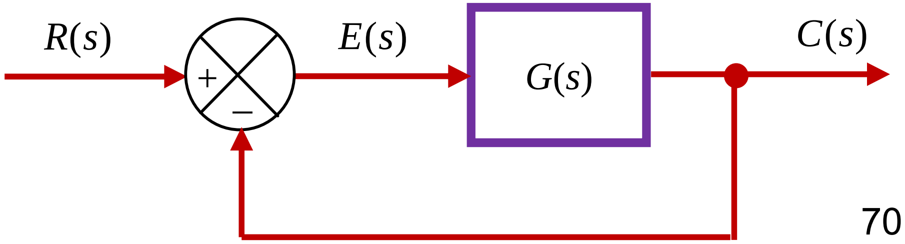
$$e_{ss} = \lim_{s\to 0} s\frac{R(s)}{1+G(s)}$$

Unit ramp input: $R(s) = \frac{1}{s^2}$

$$e_{ss} = \lim_{s\to 0} s\frac{1}{1+G(s)}\frac{1}{s^2} \Rightarrow e_{ss} = \lim_{s\to 0}\frac{1}{sG(s)} \Rightarrow e_{ss} = \frac{1}{k_v}$$

where k_v is the static velocity error constant

$$k_v = \lim_{s\to 0} sG(s)$$



Steady-State Errors in Unity-Feedback Control Systems

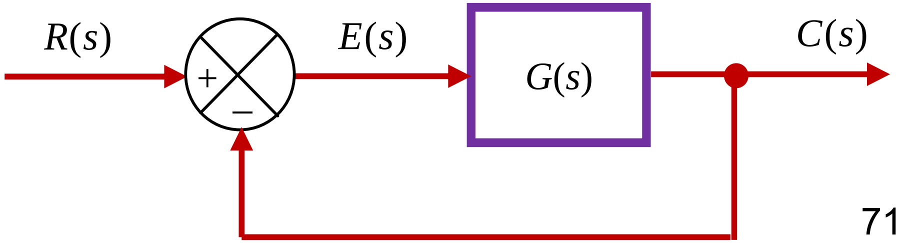
$$\boxed{e_{ss} = \lim_{s \to 0} s \frac{R(s)}{1+G(s)}}$$

Unit parabolic input: $R(s) = \dfrac{1}{s^3}$

$$e_{ss} = \lim_{s \to 0} s \frac{1}{1+G(s)} \frac{1}{s^3} \Rightarrow e_{ss} = \lim_{s \to 0} \frac{1}{s^2 G(s)} \Rightarrow \boxed{e_{ss} = \frac{1}{k_a}}$$

where k_a is the static acceleration error constant

$$k_a = \lim_{s \to 0} s^2 G(s)$$



Steady-State Errors in Unity-Feedback Control Systems

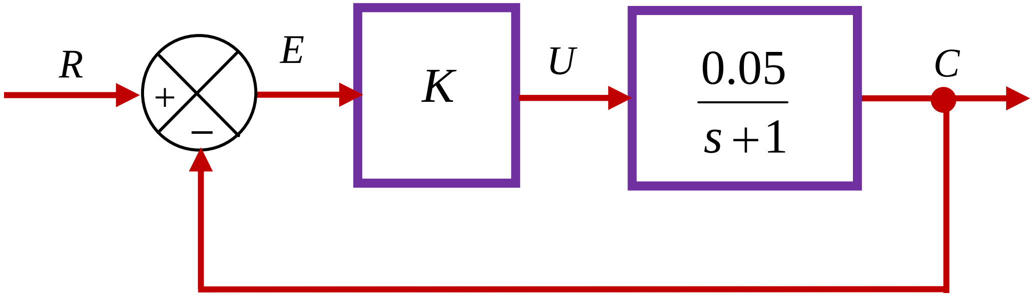
Effects of the system type on the steady-state error:

System type \ input	Unit step	Unit ramp	Unit Parabolic
0	$\frac{1}{1+k_p}$	∞	∞
1	0	$\frac{1}{k_v}$	∞
2	0	0	$\frac{1}{k_a}$
3	0	0	0

Steady-State Errors in Unity-Feedback Control Systems

Example: In the following system, calculate the gain K to have steady-state error not more than 5% in response to a unit step input.

$R(s) = \frac{1}{s}$ $\quad$ $e_{ss} \le 0.05$



$$\boxed{G(s) = \frac{0.05\ K}{1+s}}$$

$$\left.\begin{array}{l} k_p = \lim\limits_{s\to 0} G(s) = 0.05K \\ \\ e_{ss} = \dfrac{1}{1+k_p} \end{array}\right\} \Rightarrow \quad e_{ss} = \frac{1}{1+0.05\ K} \le 0.05 \quad \Rightarrow \quad K \ge 380$$