In The Name of God The Most Compassionate, The Most Merciful

Linear Control Systems

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Mathematical Model of Dynamic Systems

- A **mathematical model** of a dynamic system is defined as a set of **equations** that represents the dynamics of the system accurately or at least fairly well.
- Note that a mathematical model is **not unique** to a give system.
- A system may be represented in many **different ways** and therefore may have many mathematical models.
- The dynamics of many systems, whether they are mechanical, electrical, thermal, economic, biological and so on may be described in terms of **differential equation using physical laws**.
- For example **Newton's law** for **mechanical** systems or **Kirchhoff's laws** for **electrical** systems.

Simplicity vs. Accuracy

- **Increasing complexity** of a model can **improve the accuracy**.
- There should be **compromise** between the **simplicity** and **accuracy** of the model.
- Therefore it may be necessary to **ignore** some inherent physical **property**.
- The **ignored properties** should **not have significant effect** on the model response.

Time Domain Modelling

1. Differential equations

- For a SISO **linear time-invariant** (LTI) system, the model can be represented by the following differential equation:

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + a_2 \frac{d^{n-2} y}{dt^{n-2}} + \cdots + a_{n-1} \frac{dy}{dt} + a_n y = b_m \frac{d^m u}{dt^m} + b_{m-1} \frac{d^{m-1} u}{dt^{m-1}} + \cdots + b_1 \frac{du}{dt} + b_0 u$$

where y is the output and u is the input.

- If one or more coefficients of the above differential equation are time-dependent the system is a SISO **linear time-varying** (LTV) system:

$$\frac{d^n y}{dt^n} + a_1(t) \frac{d^{n-1} y}{dt^{n-1}} + a_2 \frac{d^{n-2} y}{dt^{n-2}} + \cdots + a_{n-1} \frac{dy}{dt} + a_n y = b_m \frac{d^m u}{dt^m} + b_{m-1} \frac{d^{m-1} u}{dt^{m-1}} + \cdots + b_1 \frac{du}{dt} + b_0 u$$

Time Domain Modelling

1. Differential equations

- The following differential equation is, for example, for a SISO **nonlinear time-invariant** system:

$$\frac{d^3y}{dt^3}+4\left(\frac{d^2y}{dt^2}\right)^2+6\frac{dy}{dt}+3y=5u$$

- The following differential equation is, for example, for a SISO **nonlinear time-varying** system:

$$\frac{d^3y}{dt^3}+4\left(\frac{d^2y}{dt^2}\right)^2+6t^3\frac{dy}{dt}+3y=5u$$

- It is impossible to represent a general differential equation for nonlinear systems.

Time Domain Modelling

1. Differential equations

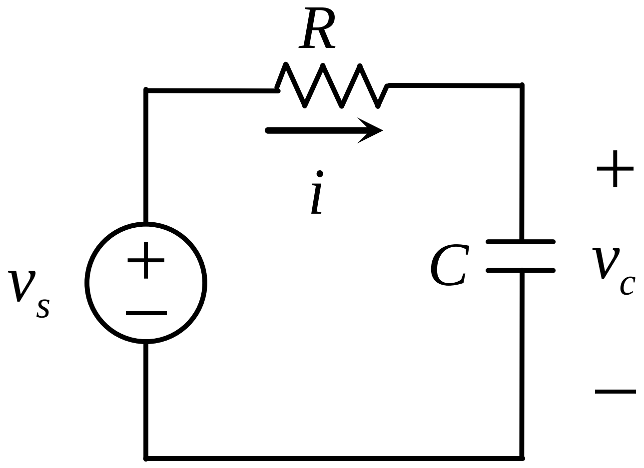
Example 1: In the following RC circuit express the relation between the input (source) voltage and the output (capacitor) voltage.

KVL:

$$Ri + v_c = v_s$$

$$RC\frac{dv_c}{dt} + v_c = v_s$$

$$\boxed{\frac{dv_c}{dt} + \frac{1}{RC}v_c = \frac{1}{RC}v_s}$$



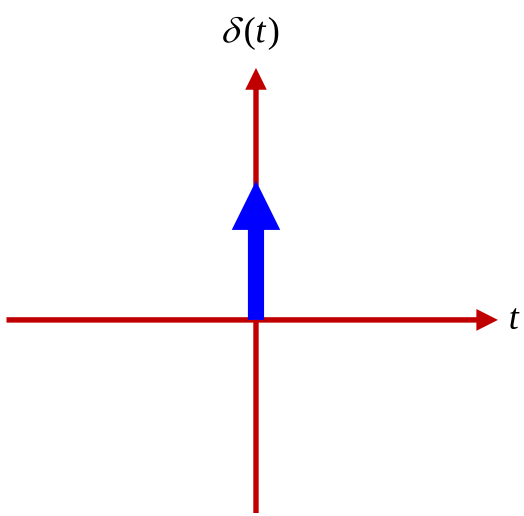
It is a linear time-invariant system.

Time Domain Modelling

1. Impulse response

- Having the impulse response, $g(t)$, the response of the system, $y(t)$, with any other input, $u(t)$, can be obtained:

$$y(t) = \int_0^t g(t-\tau)u(\tau)d\tau$$



Time Domain Modelling

3. State space equations

- **State**: The state of a dynamic system is the smallest set of variables (called state variables) such that the knowledge of these variables at $t=t_0$, together with the knowledge of the input for $t \geq t_0$, completely determines the behaviour of the system for any time $t \geq t_0$.

- **State vector**: if *n* state variables are needed to completely describe the behaviour of a given system therefore:

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^{\mathrm{T}}$$

Time Domain Modelling

3. State space equations

- **State space equations** can be defined for both linear and nonlinear with time-invariant or time-varying systems:

	Time-invariant	Time-varying
Linear	$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$ $\mathbf{y} = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$	$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$ $\mathbf{y} = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)$
Nonlinear	$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}(t), \mathbf{u}(t)\right)$ $\mathbf{y} = \mathbf{g}\left(\mathbf{x}(t), \mathbf{u}(t)\right)$	$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}(t), \mathbf{u}(t), t\right)$ $\mathbf{y} = \mathbf{g}\left(\mathbf{x}(t), \mathbf{u}(t), t\right)$

where $\mathbf{x}$ is the state vector, $\mathbf{u}$ is the input vector and $\mathbf{y}$ is the output vector

Frequency Domain Modelling

1. Frequency response

- The input to the system is a sinusoidal signal in which the frequency is variable within a range:

$$u_\omega = A\sin(\omega t) \qquad \omega \in [\omega_{\min}, \omega_{\max}]$$

- The corresponding output

$$y_\omega = B_\omega \sin(\omega t + \varphi_\omega)$$

Frequency Domain Modelling

2. Transfer function

- The transfer function of a linear, time-invariant differential equation system is defined as the ratio of the Laplace transform of the output to the Laplace transform of the input under the assumption that all initial conditions are zero:

$$G(s)=\frac{L\{y(t)\}}{L\{u(t)\}} \qquad\qquad G(s)=\frac{Y(s)}{U(s)}$$

Frequency Domain Modelling

2. Transfer function

$$L\left\{\frac{dy^n}{dt^n}\right\} = s^n Y(s)$$

- For the following differential equation which is for LTI system

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + a_2 \frac{d^{n-2} y}{dt^{n-2}} + \cdots + a_{n-1} \frac{dy}{dt} + a_n y = b_m \frac{d^m u}{dt^m} + b_{m-1} \frac{d^{m-1} u}{dt^{m-1}} + \cdots + b_1 \frac{du}{dt} + b_0 u$$

- The transfer function is as follows

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots a_{n-1} s + a_n}$$

Frequency Domain Modelling

2. Transfer function

The properties of a transfer function (TF) are as follows

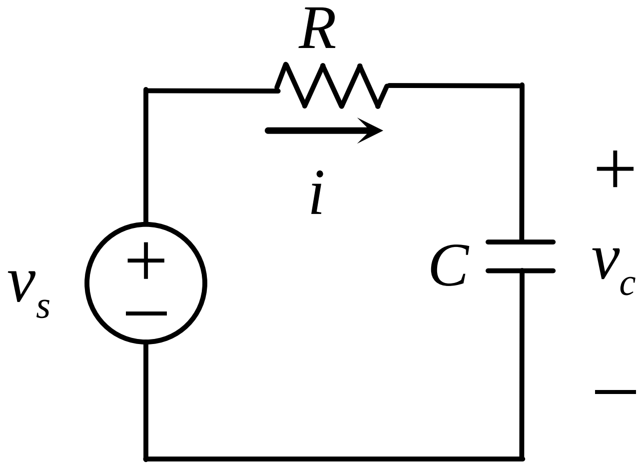
1. TF is only defined for linear time-invariant systems.
2. In TF calculation the initial values are set to zero.
3. TF is a property of a system itself, independent of the magnitude and nature of the input or driving function.
4. TF is independent of variable *t* and the only variable is *s*.

Frequency Domain Modelling

2. Transfer function

Example 2: In the following RC circuit obtain the transfer function where the input is the source voltage and the output is the capacitor voltage.

$$\frac{dv_c}{dt} + \frac{1}{RC} v_c = \frac{1}{RC} v_s$$

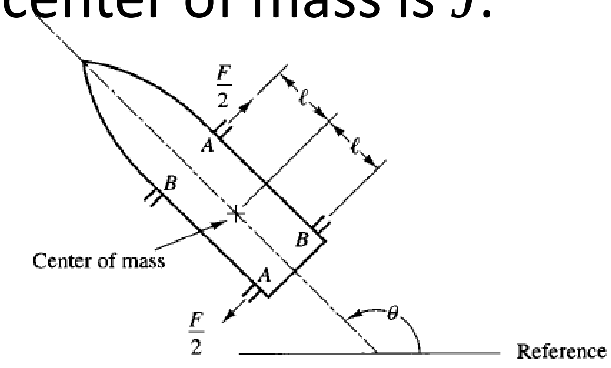


$$G(s) = \frac{V_c(s)}{V_s(s)} = \frac{\frac{1}{RC}}{s + \frac{1}{RC}}$$

Frequency Domain Modelling

2. Transfer function

Example 3: Consider the following satellite attitude control system. The diagram shows the control of only the yaw angle θ. In the actual system there are control about three axes. Small jets apply reaction forces to rotate the satellite body into the desired attitude. The two skew symmetrically placed jets denoted by A or B operate in pairs. Assume that each jet thrust is $F/2$ and a toque $T=Fl$ is applied to the system. The jets are applied for a certain time duration and thus the torque can be written as $T(t)$. The moment of inertia about the axis of rotation at the center of mass is J.



Frequency Domain Modelling

2. Transfer function

Example 3: Obtain the transfer function of this system by assuming that torque $T(t)$ is the input and the angular displacement $\theta(t)$ of the satellite is the output.

Applying Newton's second law yields :

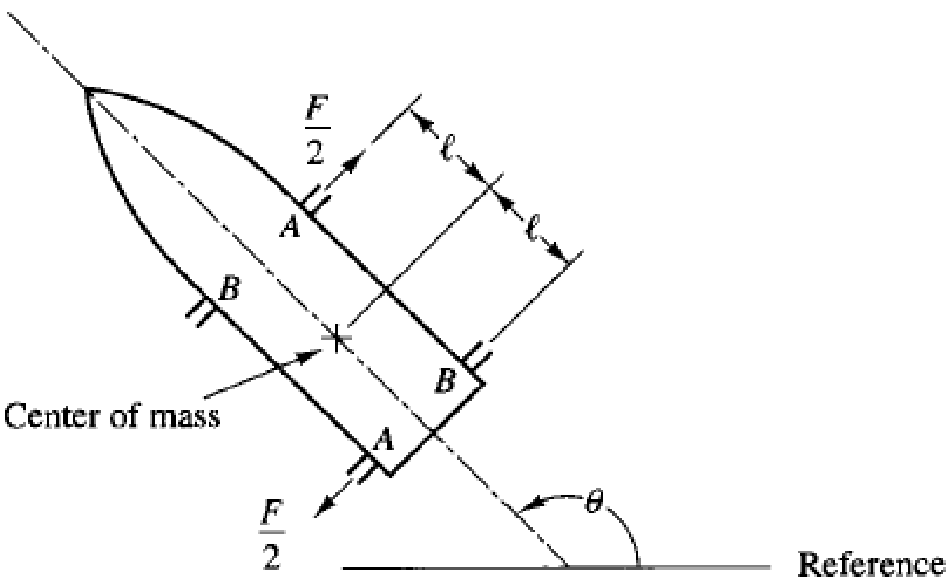
$$J\frac{d^2\theta}{dt^2} = T$$

Taking the Laplace transform

$$Js^2\Theta(s) = T(s)$$

The transfer function is

$$G(s) = \frac{\Theta(s)}{T(s)} = \frac{1}{Js^2}$$



Frequency Domain Modelling

2. Transfer function

Convolution integral: For a linear, time-invariant system the transfer function is

$$G(s) = \frac{Y(s)}{U(s)}$$

It means that the output can be written as the product of the Laplace of input and the transfer function

$$Y(s) = G(s)U(s)$$

Note that the multiplication in the complex domain is equivalent to convolution in the time domain.

$$y(t) = \int_0^t g(t-\tau)u(\tau)d\tau$$ where $g(t)$ is the impulse response.

A Point

The Laplace transfer of the impulse response is the transfer function.

Frequency domain	Time domain
Transfer function $$G(s)=\frac{Y(s)}{U(s)}$$	Impulse response $$g(t)=L^{-1}\{G(s)\}$$
$$Y(s)=G(s)U(s)$$	$$y(t)=\int_0^t g(t-\tau)u(\tau)d\tau$$

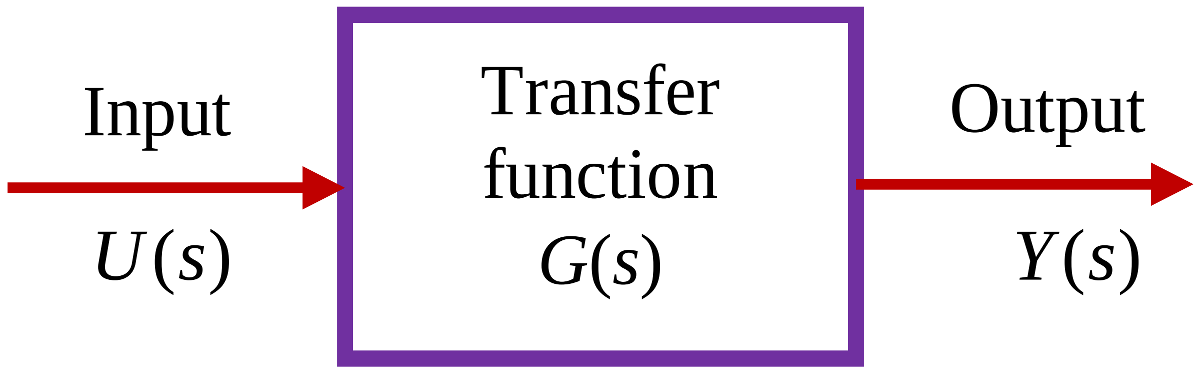
Graphical Representation

1. Block diagram

A block diagram of a system is a pictorial representation of the functions.

The arrows entering a block are inputs and those leaving a block are outputs.

Note that signal can pass only in the direction of arrows.

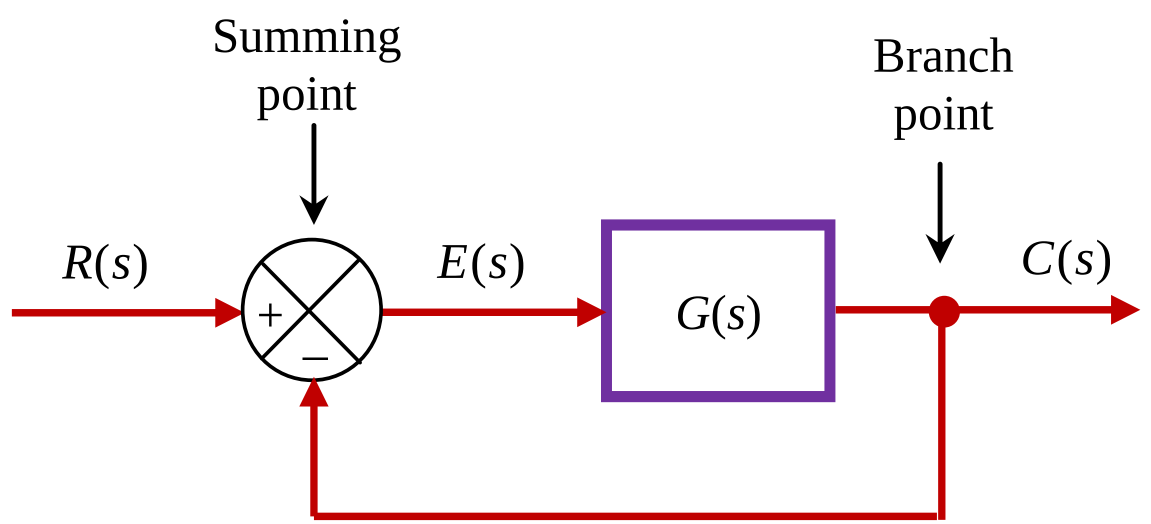


Graphical Representation

Block diagram of closed-loop systems

Summing point

Branch point



Graphical Representation

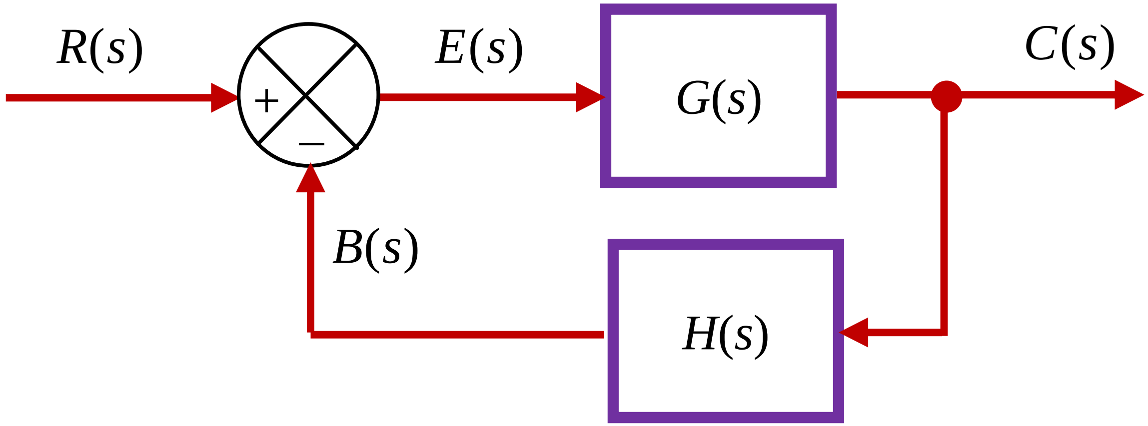
Block diagram of closed-loop systems

- **Loop transfer function** or open-loop transfer function is defined as:

$$\frac{B(s)}{E(s)} = G(s)H(s)$$

- **Feedforward transfer function** is defined as:

$$\frac{C(s)}{E(s)} = G(s)$$



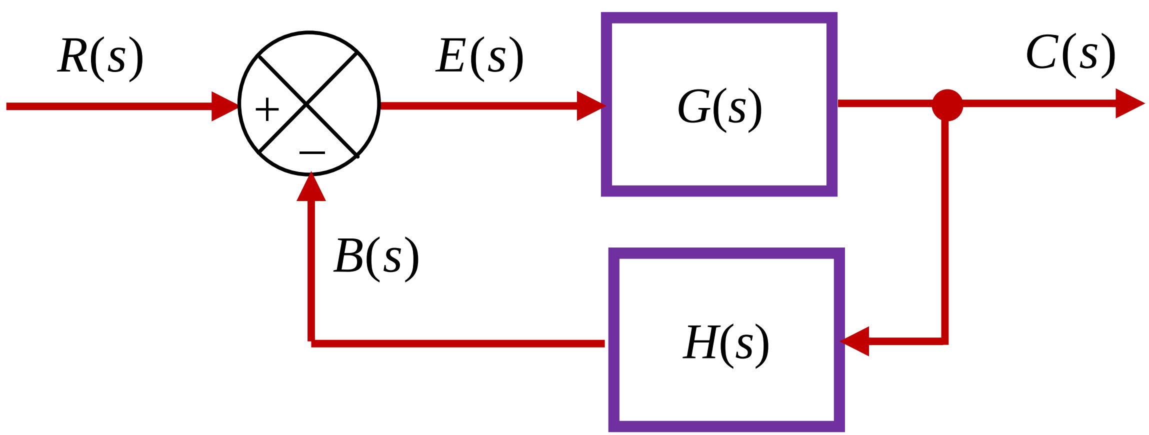
Graphical Representation

Block diagram of closed-loop systems

- **Closed-loop transfer function**: $\dfrac{C(s)}{R(s)} = ?$

$$\left.\begin{aligned} C(s) &= G(s)E(s) \\ E(s) &= R(s) - B(s) \\ &= R(s) - H(s)C(s) \end{aligned}\right\} \; C(s) = G(s)\left[R(s) - H(s)C(s)\right] \Rightarrow \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$\text{Closed loop TF} = \frac{\text{Feedforward TF}}{1+\text{Loop TF}}$$



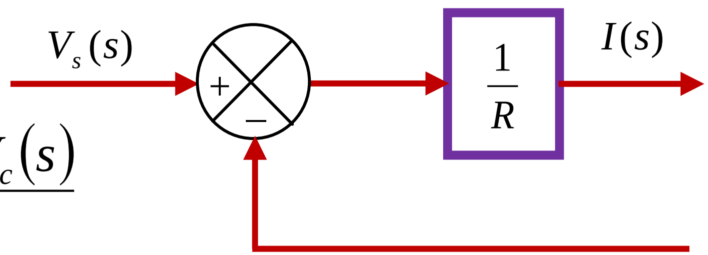
Graphical Representation

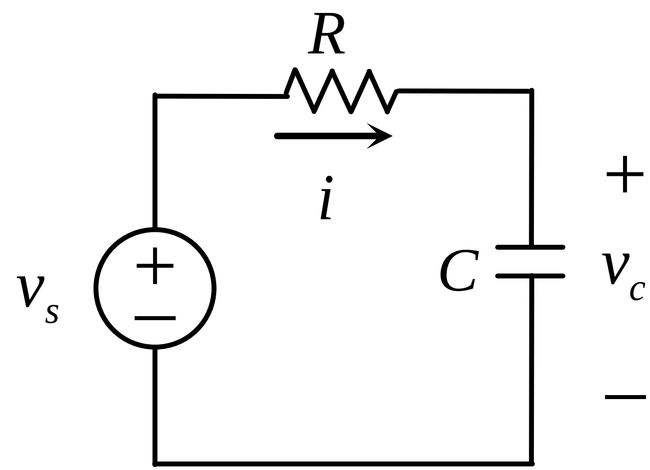
Block diagram of closed-loop systems

Example 4: Draw the block diagram of the following RC circuit

$$i=\frac{v_s-v_c}{R}$$

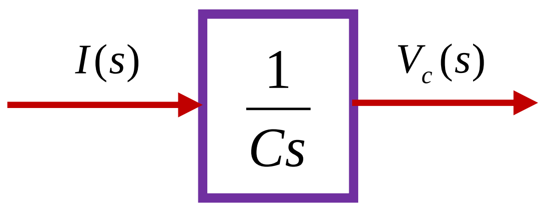
$$I(s)=\frac{V_s(s)-V_c(s)}{R}$$

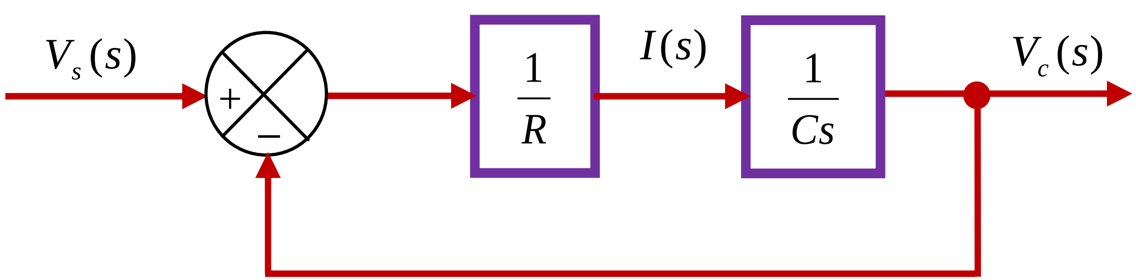




$$v_c=\frac{1}{C}\int i\,dt$$

$$V_c(s)=\frac{1}{Cs}I(s)$$

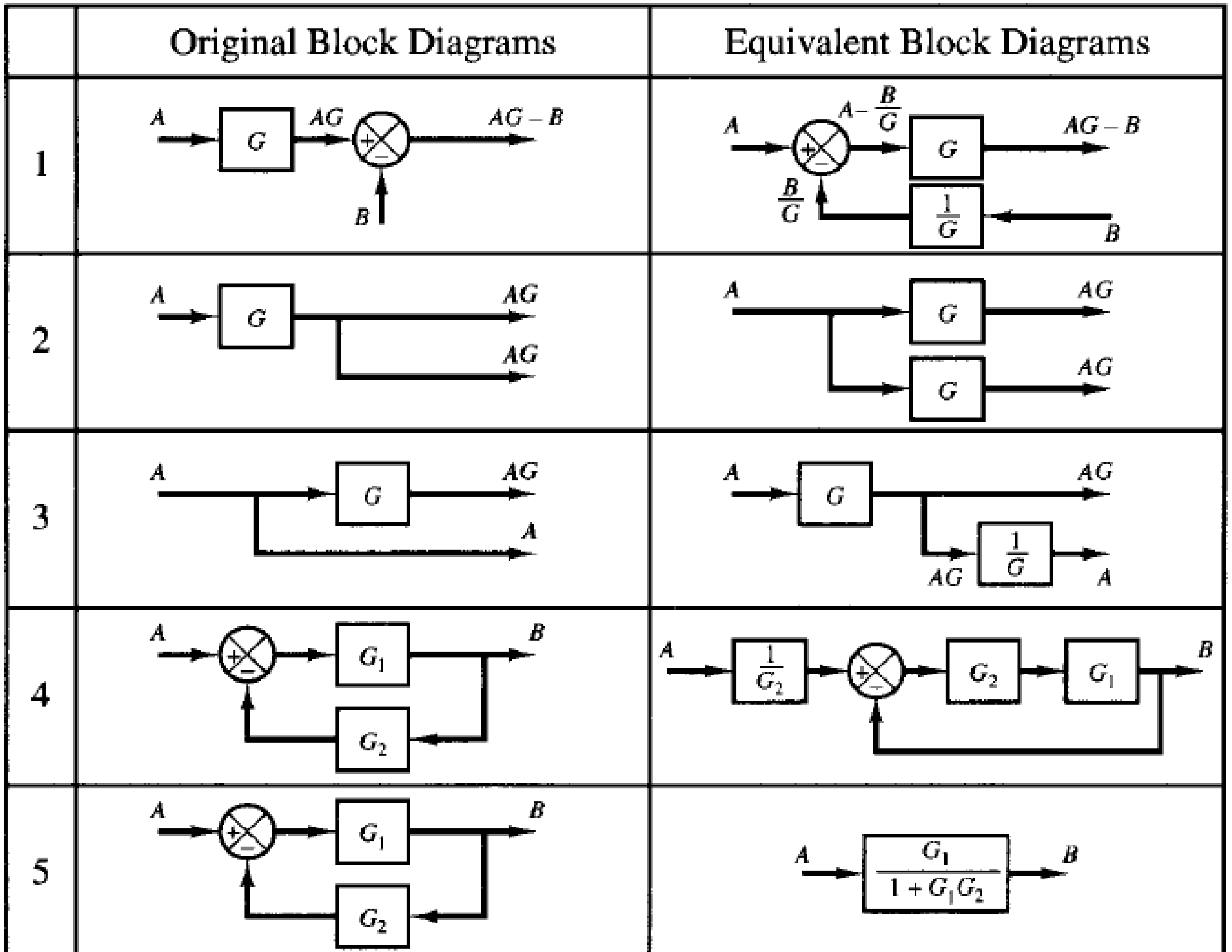




Graphical Representation

Block diagram of closed-loop systems

Rules of block diagram algebra



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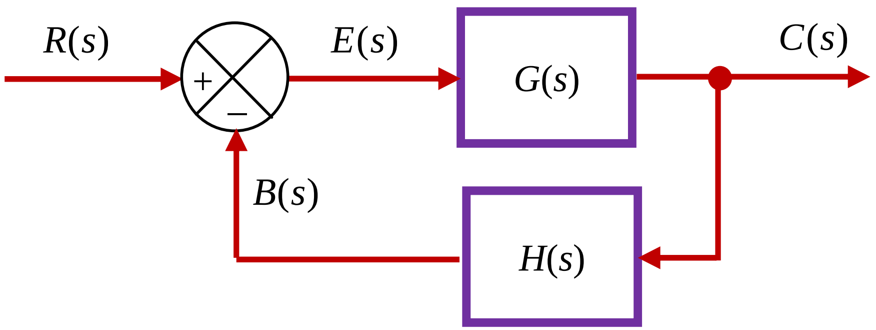
Graphical Representation

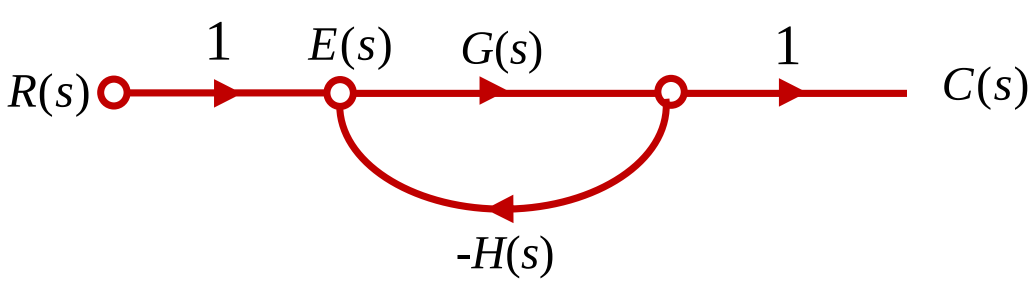
2. Signal Flow Graph (SFG)

Signal flow graph is a simpler representation of block diagram.

For example consider the following block diagram



The signal flow graph of the above block diagram is shown below

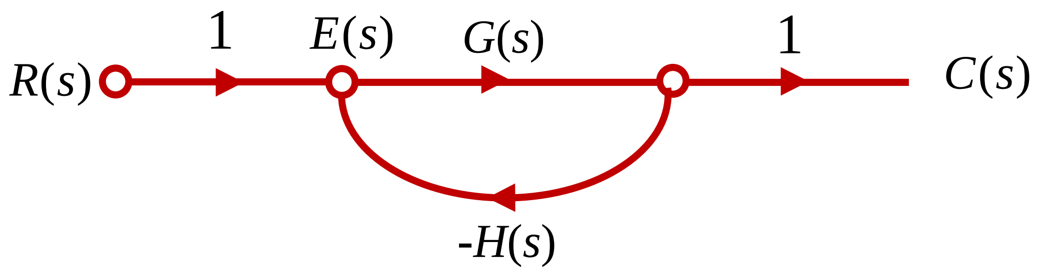


Graphical Representation

2. Signal Flow Graph (SFG)

The properties of a signal flow graph (SFG) are as follows

1. SFG is only defined for linear time-invariant systems.
2. In SFG the nodes are used as variables.
3. The flow directions are shown by arrows.
4. **Input node** is the node from which only one branch is leaving.
5. **Output node** is the node in which only one branch is entering.
6. **Forward path** is a set of branches starting from input node and ending at the output node without passing a node twice.

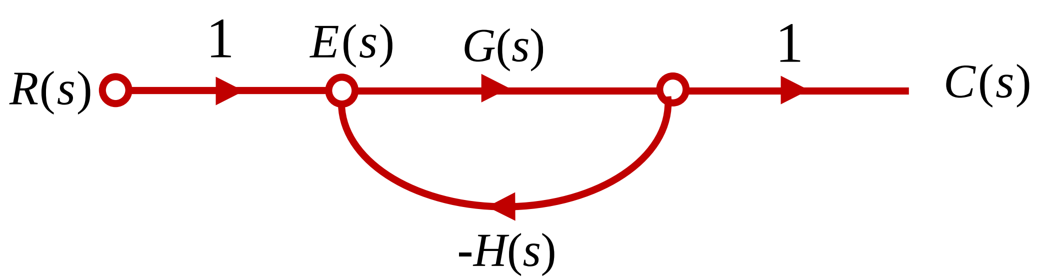


Graphical Representation

2. Signal Flow Graph (SFG)

The properties of a signal flow graph (SFG) are as follows

7. **Forward path gain** is the multiplication of the gains of the forward path branches.
8. **Loop** is a closed path starting from any node and ending at that node without passing any other node twice.
9. **Loop path gain** is the multiplication of the gains of the loop branches.



Graphical Representation

2. Signal Flow Graph (SFG)

SFG Algebra

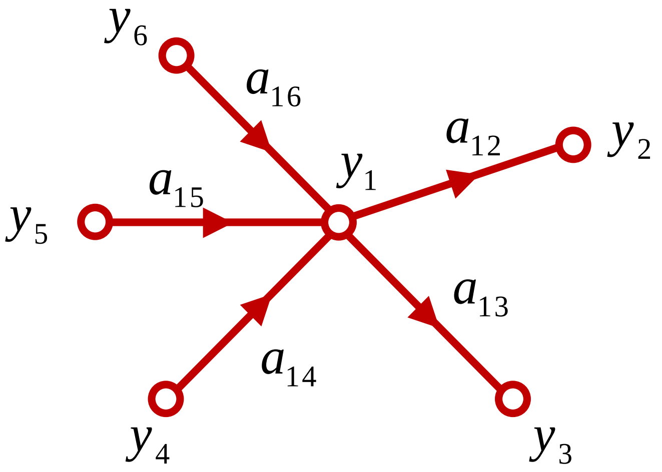
- The value dedicated to a node is equal the summation of the signals *entering* the node.

$$y_1 = a_{16}y_6 + a_{15}y_5 + a_{14}y_4$$

- The value dedicated to a node is transferred to all branches *leaving* that node.

$$y_2 = a_{12}y_1$$

$$y_3 = a_{13}y_1$$

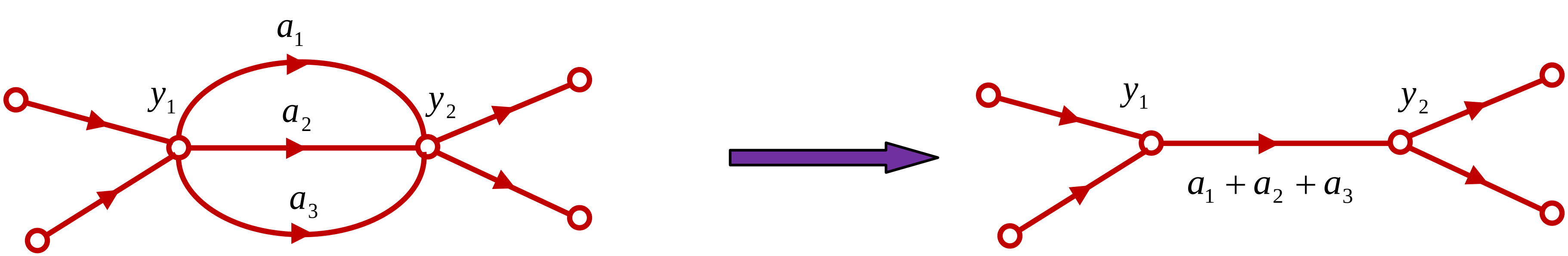


Graphical Representation

2. Signal Flow Graph (SFG)

SFG Algebra

- **Parallel connection**: The parallel branches connecting two nodes in the *same direction* can be replaced by a single branch where its gain is equal to the summation of all parallel branches:

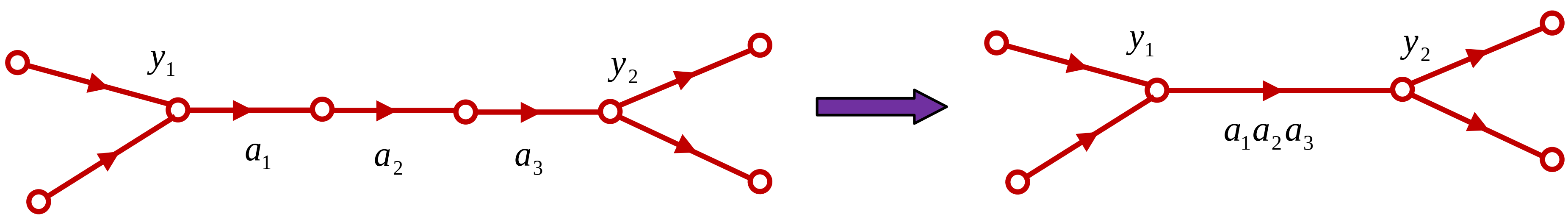


Graphical Representation

2. Signal Flow Graph (SFG)

SFG Algebra

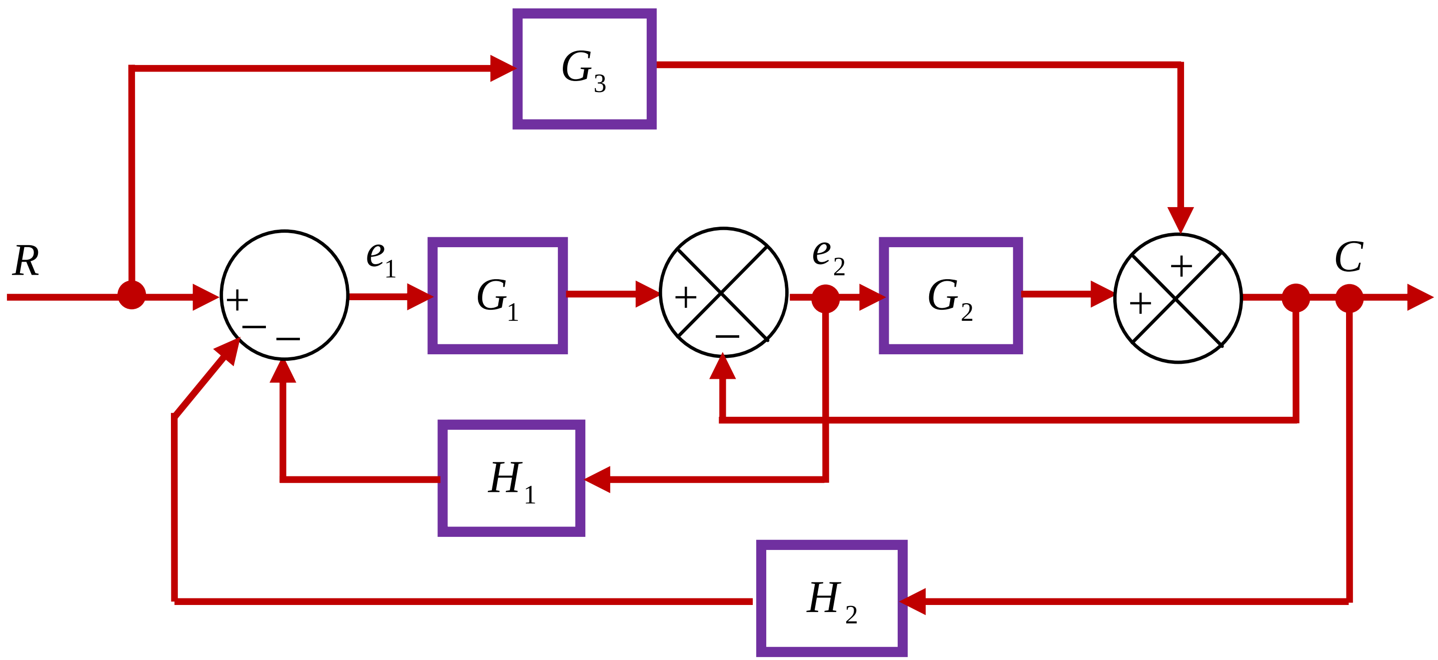
- **Series connection**: The series branches connecting two nodes in the *same direction* can be replaced by a single branch where its gain is equal to the multiplication of all series branches:



Graphical Representation

2. Signal Flow Graph (SFG)

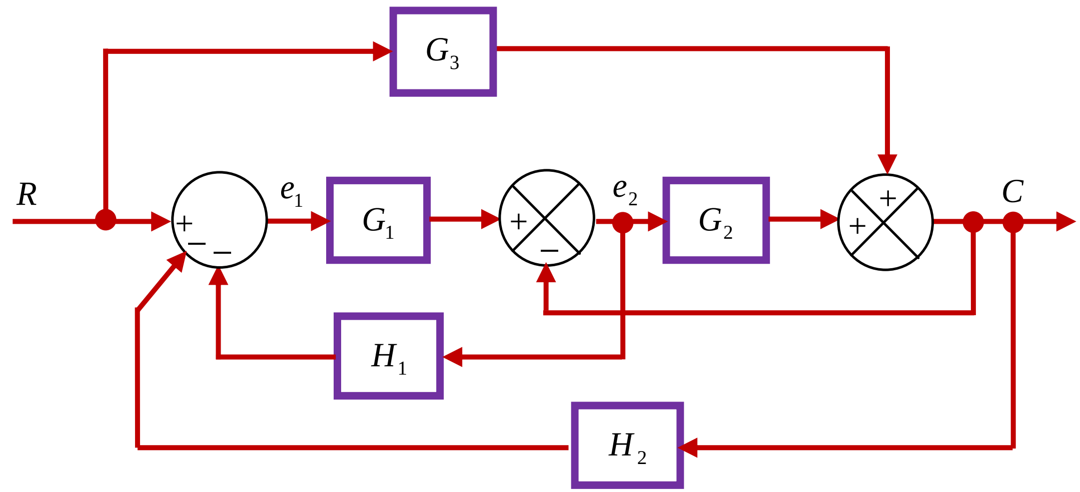
Example 5: draw the signal flow graph of the following block diagram:

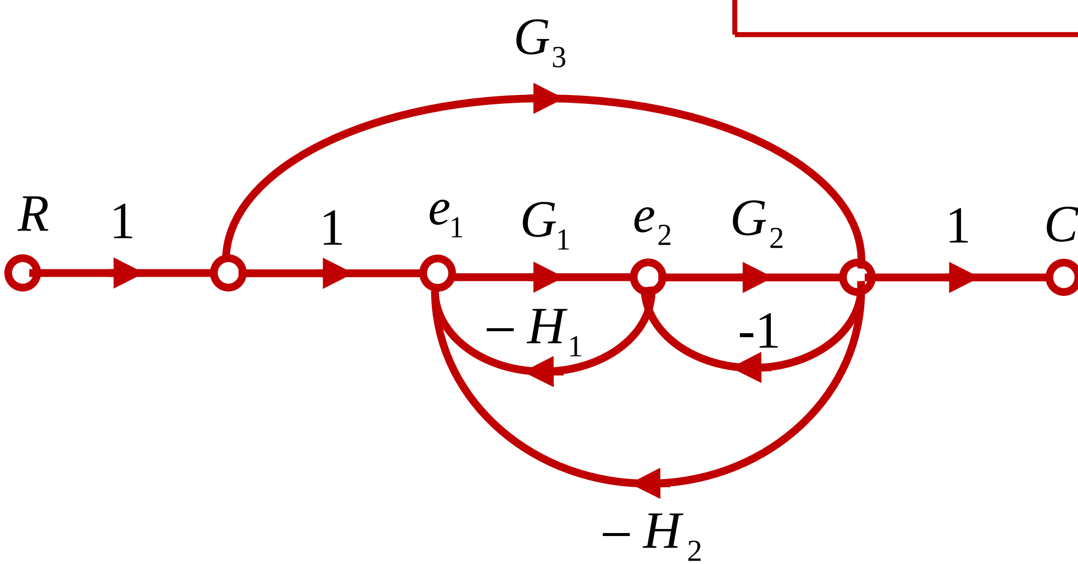


Graphical Representation

2. Signal Flow Graph (SFG)

Solution 5:





Graphical Representation

Mason’s Rule

- Mason’s rule is used to obtain the closed-loop transfer function from block diagram or signal flow graph:

$$\text{Closed}-\text{loop TF}=\frac{C(s)}{R(s)}=\frac{\sum_{k}^{N} p_k\Delta_k}{\Delta}$$

Where $\Delta = 1-\sum L_i + \sum L_i L_j - \sum L_i L_j L_l + \ldots$

p_k the k-th forward-path gain

N the number of forward paths

$\sum L_i$ the summation of all single loops

$\sum L_i L_j$ the summation of nontouching-loop gains taken 2 at a time

$\sum L_i L_j L_l$ the summation of nontouching-loop gains taken 3 at a time

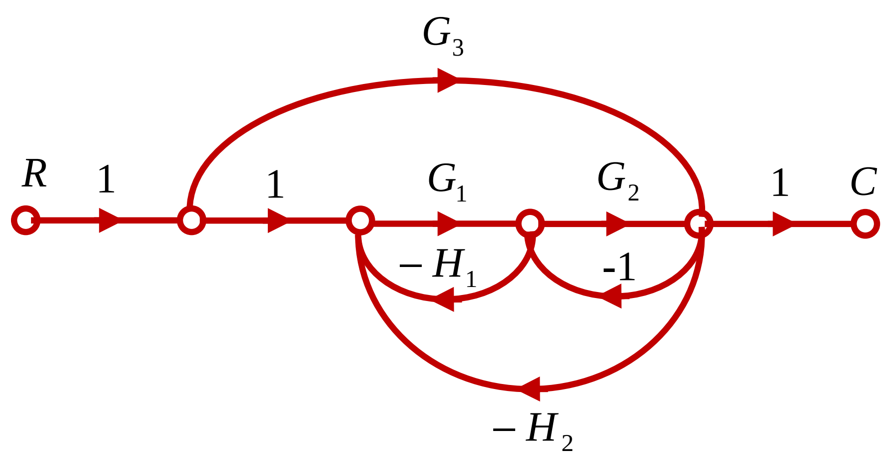
Δ_k is Δ minus the summation of the gains of the loops that touch the k-th forward path.

Graphical Representation

2. Signal Flow Graph (SFG)

- **Example 6**: find the closed-loop transfer function of the following SFG.

$$\text{Closed-loop TF} = \frac{C(s)}{R(s)} = \frac{\sum_{k}^{N} p_k \Delta_k}{\Delta}$$



There are two forward paths

$$p_1 = G_1 G_2 \qquad \Delta_1 = 1$$

$$p_2 = G_3 \qquad \Delta_2 = 1 + G_1 H_1$$

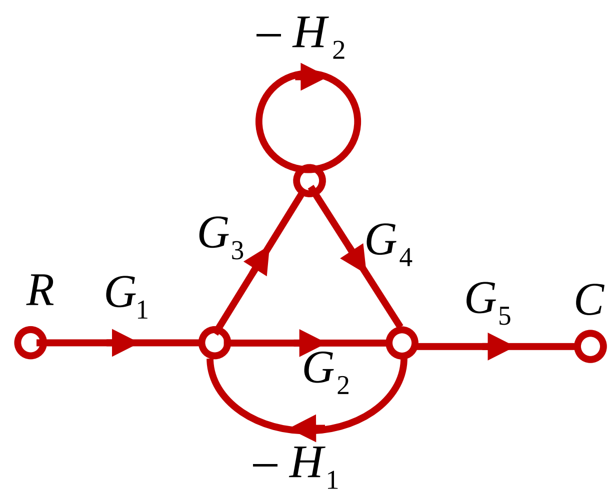
There are three loops which are all touching

$$\left.\begin{aligned} L_1 &= -G_1 H_1 \\ L_2 &= -G_2 \\ L_3 &= -G_1 G_2 H_2 \end{aligned}\right\} \Delta = 1 + G_1 H_1 + G_2 + G_1 G_2 H_2$$

$$\boxed{\frac{C(s)}{R(s)} = \frac{G_1 G_2 + G_3 + H_1 G_1 G_3}{1 + G_1 H_1 + G_2 + G_1 G_2 H_2}}$$

- **Example 7**: find the closed-loop transfer function of the following SFG.

$$\text{Closed}-\text{loop TF} = \frac{C(s)}{R(s)} = \frac{\sum_{k}^{N} p_k \Delta_k}{\Delta}$$



There are two forward paths

$p_1 = G_1 G_2 G_5$ $\qquad \Delta_1 = 1 + H_2$

$p_2 = G_1 G_3 G_4 G_5$ $\qquad \Delta_2 = 1$

There are three loops which two of them are nontouching

$L_1 = -G_2 H_1$

$L_1 L_2 = G_2 H_1 H_2$

$L_2 = -H_2$

$L_3 = -G_3 G_4 H_1$

Graphical Representation

2. Signal Flow Graph (SFG)

- **Example 7**: find the closed-loop transfer function of the following SFG.

$$\text{Closed-loop TF} = \frac{C(s)}{R(s)} = \frac{\sum_{k}^{N} p_k \Delta_k}{\Delta}$$

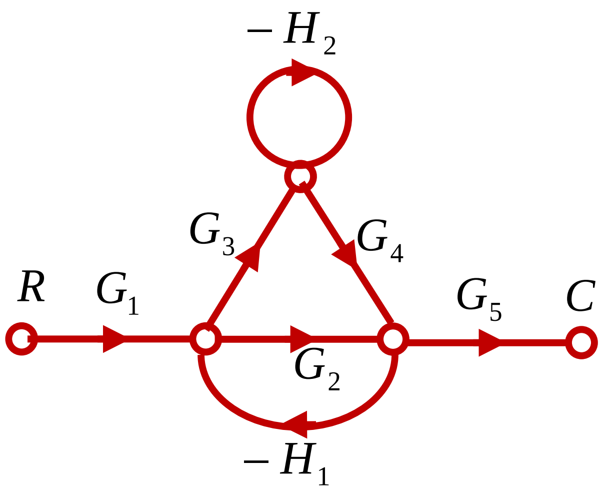
$p_1 = G_1 G_2 G_5$ $\qquad \Delta_1 = 1 + H_2$

$p_2 = G_1 G_3 G_4 G_5$ $\qquad \Delta_2 = 1$

$L_1 = -G_2 H_1$ $\qquad L_3 = -G_3 G_4 H_1$

$L_2 = -H_2$ $\qquad L_1 L_2 = G_2 H_1 H_2$

$$\frac{C(s)}{R(s)} = \frac{G_1 G_2 G_5 (1 + H_2) + G_1 G_3 G_4 G_5}{1 + G_2 H_1 + H_2 + G_3 G_4 H_1 + G_2 H_1 H_2}$$



$$\Delta = 1 + G_2 H_1 + H_2 + G_3 G_4 H_1 + G_2 H_1 H_2$$

Time Domain Modelling

State space equations

- **State**: The state of a dynamic system is the smallest set of variables (called **state variables**) such that the knowledge of these variables at $t=t_0$, together with the knowledge of the input for $t \geq t_0$, completely determines the behaviour of the system for any time $t \geq t_0$.

- **State vector**: if *n* state variables are need to completely describe the behaviour of a given system therefore:

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

Time Domain Modelling

State space equations

- **State space equations** can be defined for both linear and nonlinear with time-invariant or time-varying systems:

	Time-invariant	Time-varying
Linear	$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$ $\mathbf{y} = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$	$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$ $\mathbf{y} = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)$
Nonlinear	$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}(t), \mathbf{u}(t)\right)$ $\mathbf{y} = \mathbf{g}\left(\mathbf{x}(t), \mathbf{u}(t)\right)$	$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}(t), \mathbf{u}(t), t\right)$ $\mathbf{y} = \mathbf{g}\left(\mathbf{x}(t), \mathbf{u}(t), t\right)$

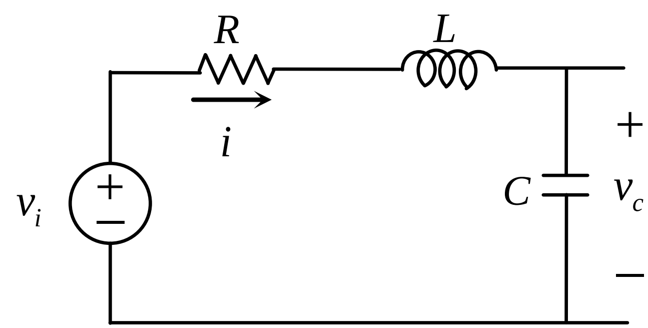
where $\mathbf{x}$ is the state vector, $\mathbf{u}$ is the input vector and $\mathbf{y}$ is the output vector

Time Domain Modelling

State space equations

- **Example 8**: Write the state space equations for the following electric circuit. The output is the capacitor voltage.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y} = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$



- The input of the system is $u = v_i$

- In electric circuits the state variables are normally defined as the current of the inductors and the voltage of the capacitors therefore: $x_1 = i$

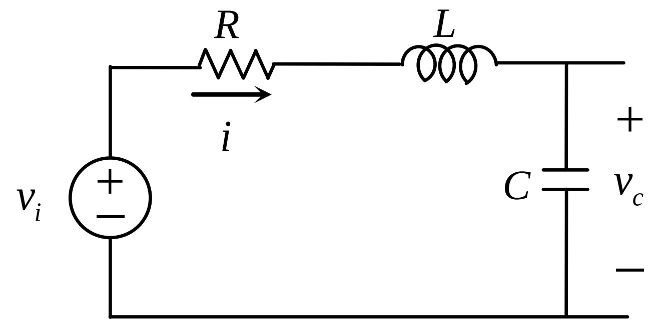
$x_2 = v_c$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} i \\ v_c \end{bmatrix}$$

- The output of the system is $y = v_c$

Time Domain Modelling

State space equations



- **Solution 8**:

KVL

$$v_i = Ri + L\frac{di}{dt} + v_c \quad \Rightarrow \quad \frac{di}{dt} = \frac{1}{L}v_i - \frac{R}{L}i - \frac{1}{L}v_c$$

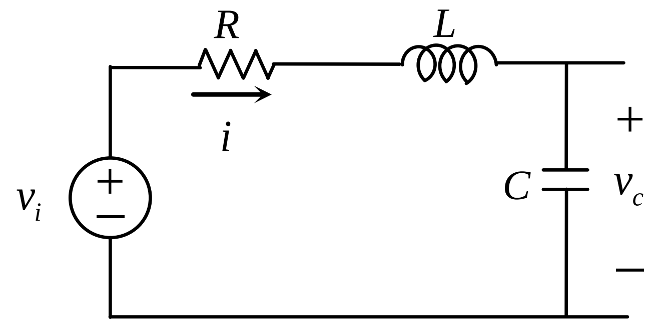
The relation between the voltage and current of the capacitor

$$i = C\frac{dv_c}{dt} \quad \Rightarrow \quad \frac{dv_c}{dt} = \frac{1}{C}i$$

Time Domain Modelling

State space equations

- **Solution 8:** $x_1 = i \qquad x_2 = v_c$



$$\begin{cases} \dfrac{di}{dt} = \dfrac{1}{L} v_i - \dfrac{R}{L} i - \dfrac{1}{L} v_c \\ \\ \dfrac{dv_c}{dt} = \dfrac{1}{C} i \end{cases}$$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\dfrac{R}{L} & -\dfrac{1}{L} \\ \dfrac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \dfrac{1}{L} \\ 0 \end{bmatrix} v_i$$

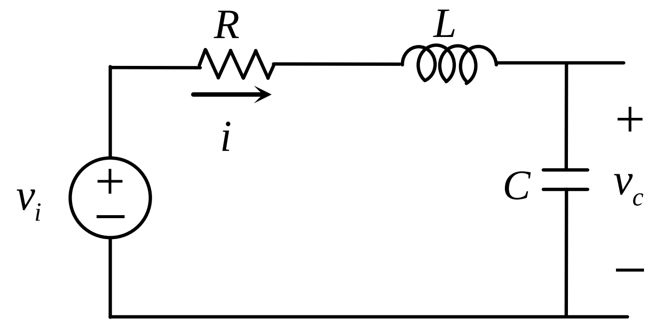
$$y = x_2$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y} = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

Time Domain Modelling

State space equations

- **Solution 8:** $x_1 = i \qquad x_2 = v_c$



$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{y} = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\dfrac{R}{L} & -\dfrac{1}{L} \\ \dfrac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \dfrac{1}{L} \\ 0 \end{bmatrix} v_i \\ y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0] v_i \end{cases}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} i \\ v_c \end{bmatrix} \qquad \dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \dfrac{di}{dt} \\ \dfrac{dv_c}{dt} \end{bmatrix}$$

$$y = v_c$$

$$\mathbf{A} = \begin{bmatrix} -\dfrac{R}{L} & -\dfrac{1}{L} \\ \dfrac{1}{C} & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} \dfrac{1}{L} \\ 0 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix} \qquad \mathbf{D} = [0]$$

Time Domain Modelling

State space equations (dimensions)

- Assume a system with n state variables, p inputs and q outputs, i.e.

$$\mathbf{x} \in \Re^{n\times 1} \qquad \mathbf{u} \in \Re^{p\times 1} \qquad \mathbf{y} \in \Re^{q\times 1}$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y} = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

- The dimensions of the matrices/vectors in the state-space equations are as follows

$\mathbf{A} \in \Re^{n\times n}$	$\mathbf{B} \in \Re^{n\times p}$
$\mathbf{C} \in \Re^{q\times n}$	$\mathbf{D} \in \Re^{q\times p}$

Time Domain Modelling

No derivative of input

Differential equations to State space equations

- Assume a system with the following differential equation:

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_{n-1} y' + a_n y = b u$$

- Note that there is **no derivative of input**.
- The corresponding state-space equations are obtained as

$$x_1 = y$$

$$x_2 = \dot{y} = \dot{x}_1$$

$$x_3 = \ddot{y} = \dot{x}_2$$

$$x_4 = \dddot{y} = \dot{x}_3$$

$$\vdots$$

$$x_{n-1} = y^{(n-2)} = \dot{x}_{n-2}$$

$$x_n = y^{(n-1)} = \dot{x}_{n-1} \Rightarrow \dot{x}_n = y^{(n)}$$

$$y^{(n)} = -a_n y - a_{n-1} y' - \cdots - a_2 y^{(n-2)} - a_1 y^{(n-1)} + b u$$

$$\Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = x_4 \\ \vdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = -a_n x_1 - a_{n-1} x_2 - \cdots - a_2 x_{n-1} - a_1 x_n + b u \end{cases}$$

Time Domain Modelling

No derivative of input

Differential equations to State space equations

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_{n-1} y' + a_n y = b u$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = x_4 \\ \vdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = -a_n x_1 - a_{n-1} x_2 - a_{n-2} x_3 - \cdots - a_2 x_{n-1} - a_1 x_n + b u \end{cases}$$

Companion form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ b \end{bmatrix} u$$

Time Domain Modelling

No derivative of input

Differential equations to State space equations

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_{n-1} y' + a_n y = bu$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ b \end{bmatrix} u$$

$x_1 = y$ $\Rightarrow$

$$y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Time Domain Modelling

No derivative of input

Differential equations to State space equations

Example 9: Obtain the state-space equation of the following differential equations

$$\frac{d^3y}{dt^3}+2\frac{d^2y}{dt^2}+5\frac{dy}{dt}-7y=3u(t)$$

$$a_1=2 \qquad a_2=5 \qquad a_3=-7 \qquad b=3$$

$$x_1=y$$
$$x_2=\dot{y}=\dot{x}_1$$
$$x_3=\ddot{y}=\dot{x}_2$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 7 & -5 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 3\end{bmatrix}u$$

$$y=\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0\end{bmatrix}u$$

$$y^{(n)}+a_1y^{(n-1)}+a_2y^{(n-2)}+\cdots+a_{n-1}y'+a_ny=bu$$

Solution of State-Space Equations

Homogeneous case: no input $\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t)$

Taking the Laplace transform yields

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\,\mathbf{X}(s) \Rightarrow s\mathbf{X}(s) - \mathbf{A}\,\mathbf{X}(s) = \mathbf{x}(0) \Rightarrow (s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{x}(0)$$

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \end{bmatrix}$$

$$\Rightarrow \boxed{\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0)} \Rightarrow \boxed{\mathbf{x}(t) = L^{-1}\left\{(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0)\right\}}$$

$L^{-1}\left\{(s\mathbf{I} - \mathbf{A})^{-1}\right\} = \varphi(t)$ is the state-transfer matrix $\Rightarrow$ $\mathbf{x}(t) = \varphi(t)\mathbf{x}(0)$

$$\varphi(t) = e^{\mathbf{A}t}$$

Solution of State-Space Equations

Inhomogeneous case: with input $\dot{\mathbf{x}}(t)=\mathbf{A}\,\mathbf{x}(t)+\mathbf{B}\,\mathbf{u}(t)$

Taking the Laplace transform yields

$$s\mathbf{X}(s)-\mathbf{x}(0)=\mathbf{A}\,\mathbf{X}(s)+\mathbf{B}\mathbf{U}(s) \Rightarrow (s\mathbf{I}-\mathbf{A})\mathbf{X}(s)=\mathbf{x}(0)+\mathbf{B}\,\mathbf{U}(s)$$

$$\Rightarrow \mathbf{X}(s)=(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{x}(0)+(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)$$

$$\Rightarrow \mathbf{x}(t)=L^{-1}\left\{(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{x}(0)+(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\,\mathbf{U}(s)\right\}$$

$$\Rightarrow \mathbf{x}(t)=\varphi(t)\mathbf{x}(0)+\int_0^t \varphi(t-\tau)\mathbf{B}\,u(\tau)\,d\tau$$

$$L^{-1}\{F_1(s)F_2(s)\}=\int_0^t f_1(t-\tau)\,f_2(\tau)\,d\tau$$

Solution of State-Space Equations

Example 10: In the following equations obtain x_1 and x_2 if the input is a unit step.

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u(t) \qquad x_1(0)=-1 \qquad x_2(0)=0$$

Unit step $\quad U(s)=\dfrac{1}{s}$

$$\mathbf{X}(s)=(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{x}(0)+(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)$$

$$\mathbf{x}(t)=L^{-1}\left\{(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{x}(0)+(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\,\mathbf{U}(s)\right\}$$

Solution of State-Space Equations

Solution 10: $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u(t)$ $\quad \mathbf{x}(0)=\begin{bmatrix}-1\\ 0\end{bmatrix}$ $\quad U(s)=\dfrac{1}{s}$

$$\mathbf{X}(s) = (s\mathbf{I}-\mathbf{A})^{-1}\mathbf{x}(0) + (s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)$$

$$(s\mathbf{I}-\mathbf{A})= s\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}-\begin{pmatrix}0 & 1\\ -2 & -3\end{pmatrix}=\begin{pmatrix}s & -1\\ 2 & s+3\end{pmatrix}$$

$$(s\mathbf{I}-\mathbf{A})^{-1}=\frac{1}{s^2+3s+2}\begin{pmatrix}s+3 & 1\\ -2 & s\end{pmatrix}=\begin{pmatrix}\dfrac{s+3}{(s+1)(s+2)} & \dfrac{1}{(s+1)(s+2)}\\ \dfrac{-2}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)}\end{pmatrix}$$

$$(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{x}(0)=\begin{pmatrix}\dfrac{s+3}{(s+1)(s+2)} & \dfrac{1}{(s+1)(s+2)}\\ \dfrac{-2}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)}\end{pmatrix}\begin{pmatrix}-1\\ 0\end{pmatrix}=\begin{pmatrix}-\dfrac{s+3}{(s+1)(s+2)}\\ \dfrac{2}{(s+1)(s+2)}\end{pmatrix}$$

Solution of State-Space Equations

Solution 10: $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \qquad \mathbf{x}(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \qquad U(s) = \frac{1}{s}$

$$\mathbf{X}(s) = (s\mathbf{I}-\mathbf{A})^{-1}\mathbf{x}(0) + (s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}U(s)$$

$$(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} = \begin{pmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{(s+1)(s+2)} \\ \frac{s}{(s+1)(s+2)} \end{pmatrix}$$

$$(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}U(s) = \begin{pmatrix} \frac{1}{s(s+1)(s+2)} \\ \frac{1}{(s+1)(s+2)} \end{pmatrix}$$

Solution of State-Space Equations

Solution 10: $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \qquad \mathbf{x}(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \qquad U(s) = \frac{1}{s}$

$$\mathbf{X}(s) = (s\mathbf{I}-\mathbf{A})^{-1}\mathbf{x}(0) + (s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}U(s)$$

$$\mathbf{X}(s) = \begin{pmatrix} \dfrac{-(s+3)}{(s+1)(s+2)} \\ \dfrac{2}{(s+1)(s+2)} \end{pmatrix} + \begin{pmatrix} \dfrac{1}{s(s+1)(s+2)} \\ \dfrac{1}{(s+1)(s+2)} \end{pmatrix} = \begin{pmatrix} \dfrac{-s^2-3s+1}{s(s+1)(s+2)} \\ \dfrac{3}{(s+1)(s+2)} \end{pmatrix} = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}$$

Using partial fraction expansion

$$G_1 = \frac{-s^2-3s+1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} = \frac{0.5}{s} + \frac{-3}{s+1} + \frac{1.5}{s+2}$$

$$G_2 = \frac{3}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} = \frac{3}{s+1} + \frac{-3}{s+2}$$

Solution of State-Space Equations

Solution 10: $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u(t)$ $\qquad \mathbf{x}(0)=\begin{bmatrix}-1\\ 0\end{bmatrix}$ $\qquad U(s)=\frac{1}{s}$

$$\mathbf{X}(s) = (s\mathbf{I}-\mathbf{A})^{-1}\mathbf{x}(0) + (s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}U(s)$$

$$\mathbf{X}(s)=\begin{pmatrix}\frac{0.5}{s}-\frac{3}{s+1}+\frac{1.5}{s+2}\\ \frac{3}{s+1}-\frac{3}{s+2}\end{pmatrix}$$

$$L^{-1}\left\{\frac{b}{s}\right\}=b\,u(t)$$

$$L^{-1}\left\{\frac{b}{s+a}\right\}=b\,e^{-at}u(t)$$

$$\mathbf{x}(t)=L^{-1}\left\{(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{x}(0)+(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\,\mathbf{U}(s)\right\}$$

$$\mathbf{x}(t)=\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix}=\begin{pmatrix}\left(0.5-3e^{-t}+1.5e^{-2t}\right)u(t)\\ \left(3e^{-t}-3e^{-2t}\right)u(t)\end{pmatrix}$$

With derivatives of input

Time Domain Modelling

Differential equations to State space equations

- Assume a system with the following differential equation:

$$y^{(n)}+a_1 y^{(n-1)}+a_2 y^{(n-2)}+\cdots+a_{n-1}y'+a_n y=b_0\,u^{(n)}+b_1\,u^{(n-1)}+\cdots+b_{n-1}u'+b_n u$$

- Note that there are **derivatives of input**.
- The corresponding state-space equations are obtained as

$$y=x_1+\beta_0 u \;\Rightarrow\; x_1=y-\beta_0 u \;\Rightarrow\; \dot{x}_1=\dot{y}-\beta_0\dot{u} \;\Rightarrow\; \dot{x}_1=x_2+\beta_1 u$$

$$x_2=\dot{y}-\beta_0\dot{u}-\beta_1 u$$

$$x_3=\ddot{y}-\beta_0\ddot{u}-\beta_1\dot{u}-\beta_2 u$$

$$\Rightarrow\; \dot{x}_2=x_3+\beta_2 u$$

$$\vdots$$

$$x_{n-1}=y^{(n-2)}-\beta_0 u^{(n-2)}-\beta_1 u^{(n-3)}-\cdots-\beta_{n-3}\dot{u}-\beta_{n-2}u \;\Rightarrow\; \dot{x}_{n-1}=x_n+\beta_{n-1}u$$

$$x_n=y^{(n-1)}-\beta_0 u^{(n-1)}-\beta_1 u^{(n-2)}-\cdots-\beta_{n-2}\dot{u}-\beta_{n-1}u \;\Rightarrow\; \dot{x}_n=?$$

Time Domain Modelling

With derivatives of input

Differential equations to State space equations

To obtain the original differential equation the following relations should be considered:

$$\beta_0 = b_0$$

$$\beta_1 = b_1 - \beta_0 a_1$$

$$\beta_2 = b_2 - \beta_1 a_1 - \beta_0 a_2$$

$$\beta_3 = b_3 - \beta_2 a_1 - \beta_1 a_2 - \beta_0 a_3$$

$$\vdots$$

$$\beta_{n-1} = b_{n-1} - \beta_{n-2} a_1 - \cdots - \beta_1 a_{n-2} - \beta_0 a_{n-1}$$

$$\beta_n = b_n - \beta_{n-1} a_1 - \cdots - \beta_1 a_{n-1} - \beta_0 a_n$$

The last equation is now obtained

$$\dot{x}_n = -a_n x_1 - a_{n-1} x_2 - \cdots - a_2 x_{n-1} - a_1 x_n + \beta_n u$$

With derivatives of input

Time Domain Modelling

Differential equations to State space equations

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_{n-1} y' + a_n y = b_0\, u^{(n)} + b_1\, u^{(n-1)} + \cdots + b_{n-1} u' + b_n u$$

$$\begin{cases} \dot{x}_1 = x_2 + \beta_1 u \\ \dot{x}_2 = x_3 + \beta_2 u \\ \dot{x}_3 = x_4 + \beta_3 u \\ \vdots \\ \dot{x}_{n-1} = x_n + \beta_{n-1} u \\ \dot{x}_n = -a_n x_1 - a_{n-1} x_2 - a_{n-2} x_3 - \cdots - a_2 x_{n-1} - a_1 x_n + \beta_n u \end{cases}$$

Companion form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \beta_{n-1} \\ \beta_n \end{bmatrix} u$$

Time Domain Modelling

With derivatives of input

Differential equations to State space equations

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_{n-1} y' + a_n y = b_0 u^{(n)} + b_1 u^{(n-1)} + \cdots + b_{n-1} u' + b_n u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \beta_{n-1} \\ \beta_n \end{bmatrix} u$$

$$y = x_1 + \beta_0 u \quad \Rightarrow \quad y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \beta_0 u$$

State-Space Equations

Example 11: Represent the state-space equations in the companion form for the following transfer function.

$$G(s) = \frac{2s+3}{s^3+4s+2}$$

First let express the differential equation

$$G(s) = \frac{Y(s)}{U(s)} = \frac{2s+3}{s^3+4s+2} \Rightarrow \frac{d^3y}{dt^3} + 4\frac{dy}{dt} + 2y = 2\frac{du}{dt} + 3u$$

$a_1 = 0$ $a_2 = 4$ $a_3 = 2$ $b_0 = 0$ $b_1 = 0$ $b_2 = 2$ $b_3 = 3$

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_{n-1} y' + a_n y = b_0 u^{(n)} + b_1 u^{(n-1)} + \cdots + b_{n-1} u' + b_n u$$

State-Space Equations

Solution 11:

$$\frac{d^3y}{dt^3}+4\frac{dy}{dt}+2y=2\frac{du}{dt}+3u$$

$a_1=0$ $\quad a_2=4$ $\quad a_3=2$ $\quad b_0=0$ $\quad b_1=0$ $\quad b_2=2$ $\quad b_3=3$

$$\beta_0=b_0=0$$

$$\beta_1=b_1-\beta_0a_1=0$$

$$\beta_2=b_2-\beta_1a_1-\beta_0a_2=2$$

$$\beta_3=b_3-\beta_2a_1-\beta_1a_2-\beta_0a_3=3$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -2 & -4 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0\\ 2\\ 3\end{bmatrix}u$$

$$y=\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+[0]u$$

State-Space Equations to Transfer Function

$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \quad \Rightarrow \quad (s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \cancelto{0}{\mathbf{x}(0)} + \mathbf{B}\,\mathbf{U}(s)$ (1)

$\mathbf{y} = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \quad \Rightarrow \quad \mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\,\mathbf{U}(s)$ (2)

---

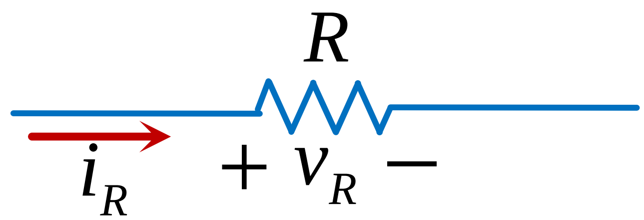
(1) $\Rightarrow \quad \mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)$ (3)

(2) & (3) $\Rightarrow \quad \mathbf{Y}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) + \mathbf{D}\,\mathbf{U}(s)$

$$\Rightarrow \quad \boxed{\frac{\mathbf{Y}(s)}{\mathbf{U}(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}}$$

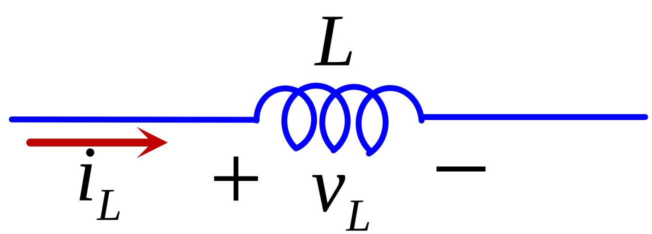
Modelling of Electrical Systems

- The fundamental law governing the electrical system is KVL & KCL.
- **Resistors**



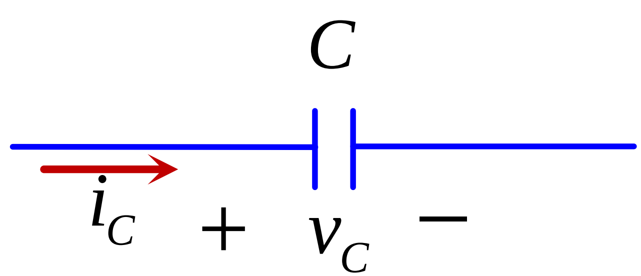
$$v_R(t) = R\, i_R(t)$$

- **Inductors**



$$v_L(t) = L\,\frac{d\, i_L}{dt}$$

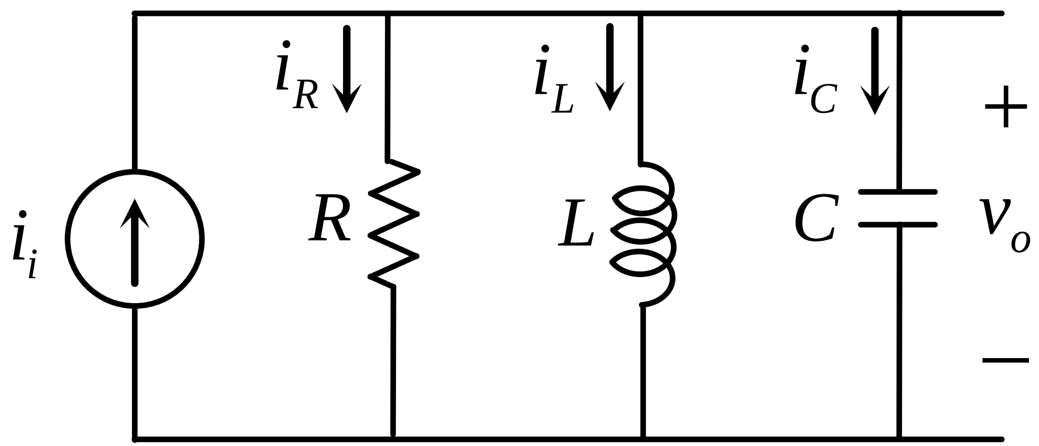
- **Capacitors**



$$i_C(t) = C\,\frac{d\, v_C}{dt}$$

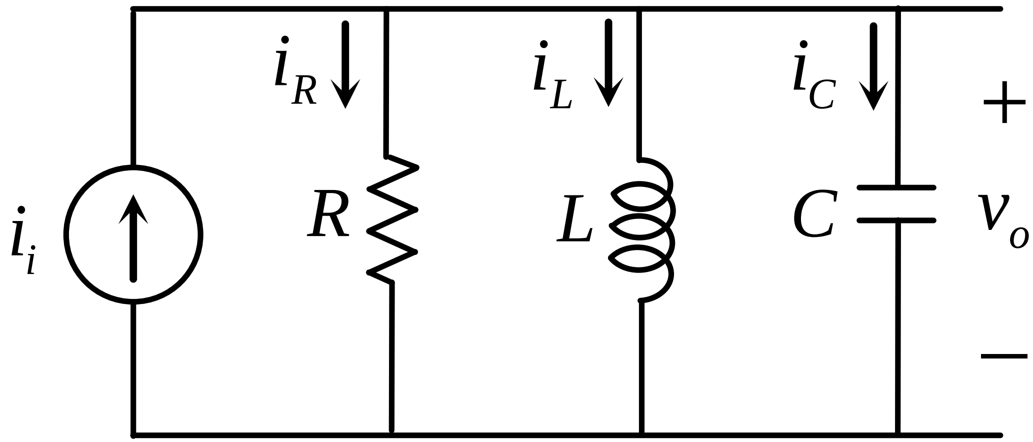
Modelling of Electrical Systems

- **Example 12:** Obtain the transfer function of the following electric circuit if the voltage is the output and the current source is the input.



Modelling of Electrical Systems

- **Solution 12:**



KCL

$$i_i = i_R + i_L + i_C \quad \Rightarrow \quad i_i = \frac{v_o}{R} + \frac{1}{L}\int_0^t v_o dt + C\frac{dv_o}{dt}$$

Taking the Laplace transform

$$I_i(s) = \left(\frac{1}{R} + \frac{1}{Ls} + Cs\right)V_o(s) \quad \Rightarrow \quad \boxed{\frac{V_o(s)}{I_i(s)} = \frac{s}{Cs^2 + \frac{1}{R}s + \frac{1}{L}}}$$

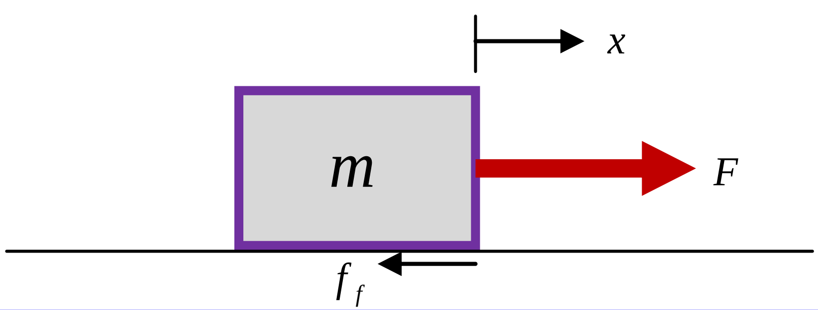
Modelling of Mechanical Systems

- The fundamental law governing the mechanical system is Newton's second law. $\sum F = ma$

- Assume a force, F, is exerted on a body with mass of m that causes the body moves on a surface with viscous friction coefficient β.
- The mathematical model of the system can be expressed as

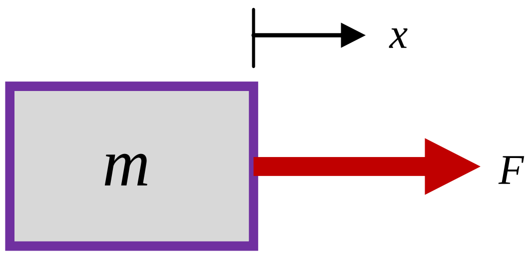
$$\sum F = ma \Rightarrow F - f_f = m\ddot{x} \Rightarrow F - \beta\dot{x} = m\ddot{x}$$

$$\Rightarrow F = m\ddot{x} + \beta\dot{x} \Rightarrow F(s) = \left(ms^2 + \beta s\right)X(s) \Rightarrow \frac{X(s)}{F(s)} = \frac{1}{s(ms+\beta)}$$



Mechanical Elements

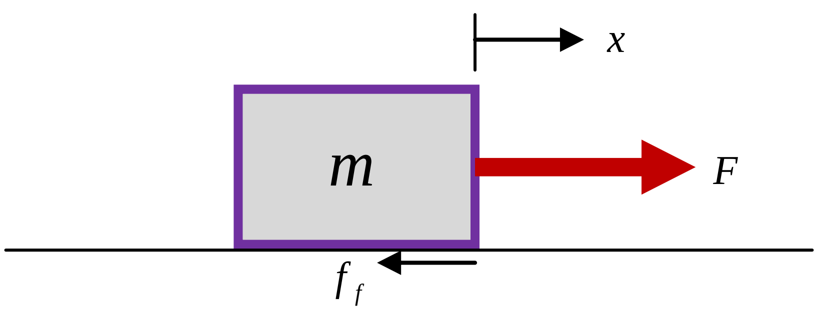
- **Mass without friction**



$$F = m\ddot{x}$$

Where m is the body mass.

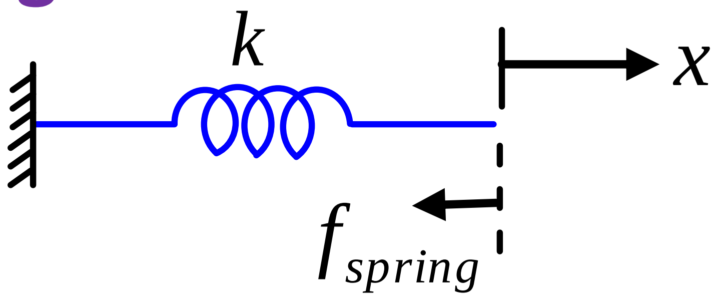
- **Mass with friction**



$$F = m\ddot{x} + \beta\dot{x}$$

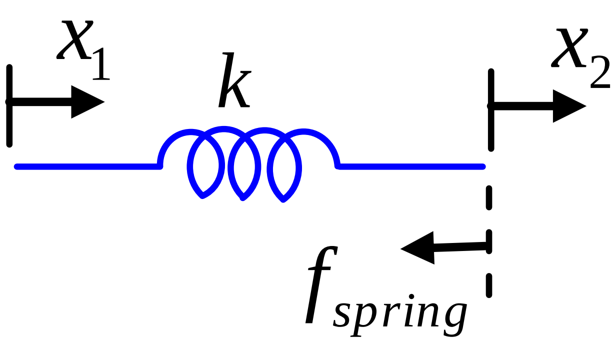
Mechanical Elements

- **Spring**



$$f_{spring} = k\,x$$

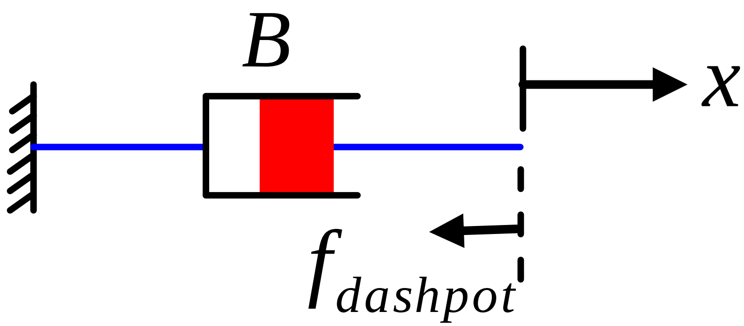
Where k is the spring constant.



$$f_{spring} = k\left(x_2 - x_1\right)$$

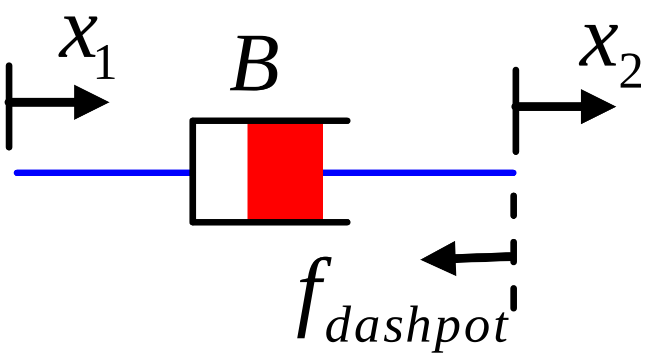
Mechanical Elements

- **Dashpot**



$$f_{dashpot} = B\dot{x}$$

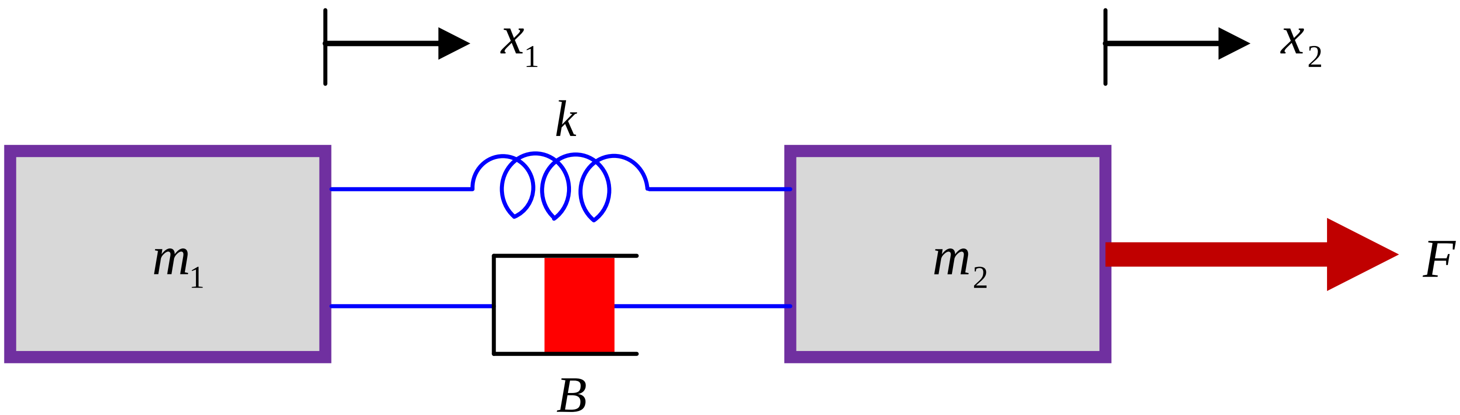
Where B is the dashpot viscous friction coefficient.



$$f_{dashpot} = B\left(\dot{x}_2 - \dot{x}_1\right)$$

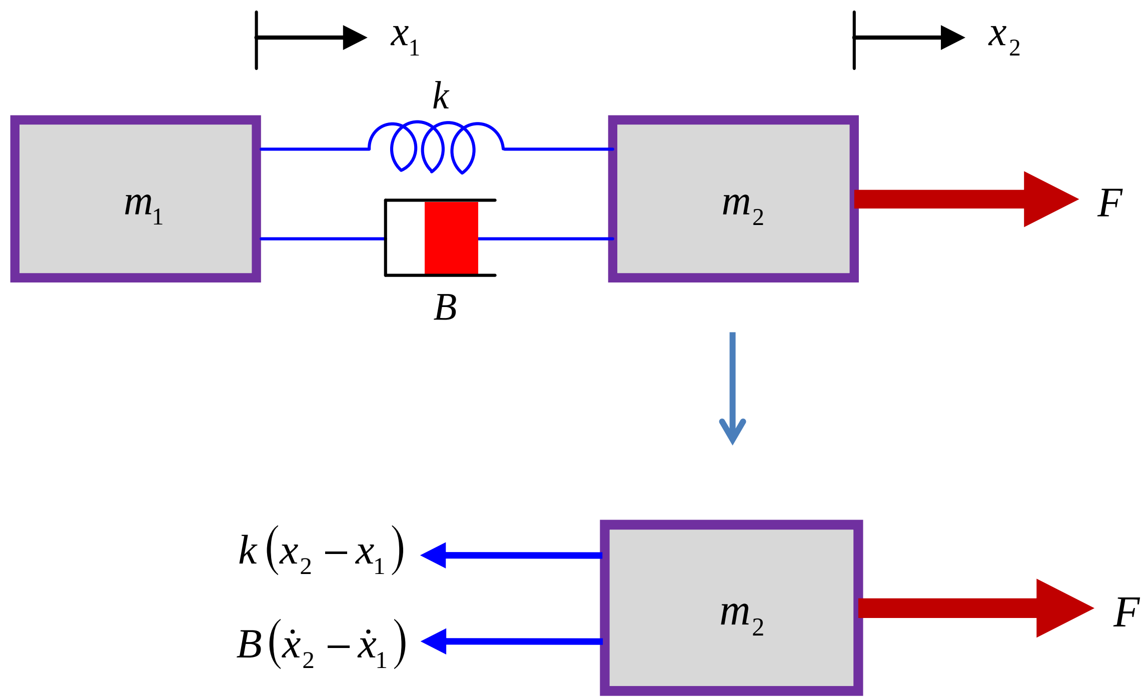
Modelling of Mechanical Systems

Example 13: Obtain the governing dynamic equation of the following system in which the bodies move on a frictionless surface.



Modelling of Mechanical Systems

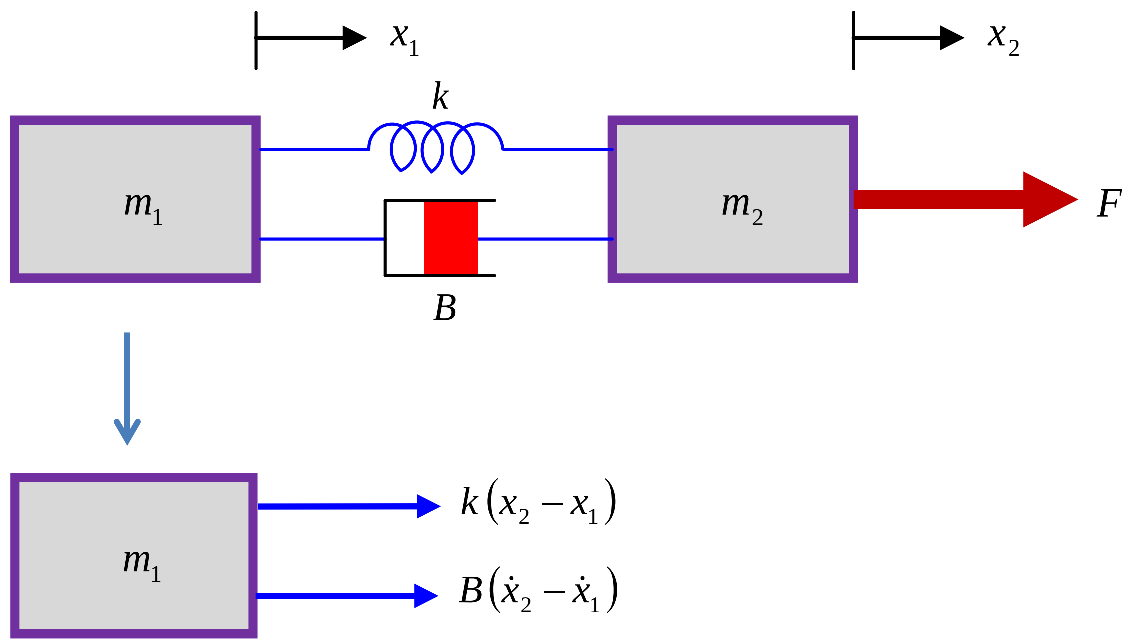
Solution 13:



$$\sum F = m_2\,\ddot{x}_2 \quad \Rightarrow \quad F - k(x_2 - x_1) - B(\dot{x}_2 - \dot{x}_1) = m_2\,\ddot{x}_2 \qquad (1)$$

Modelling of Mechanical Systems

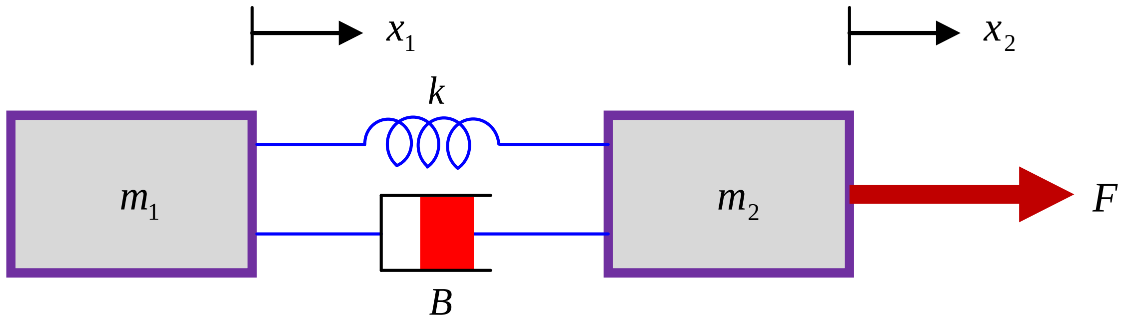
Solution 13:



$$\sum F = m_1 \ddot{x}_1 \quad \Rightarrow \quad k(x_2 - x_1) + B(\dot{x}_2 - \dot{x}_1) = m_1 \ddot{x}_1 \qquad (2)$$

Modelling of Mechanical Systems

Solution 13:



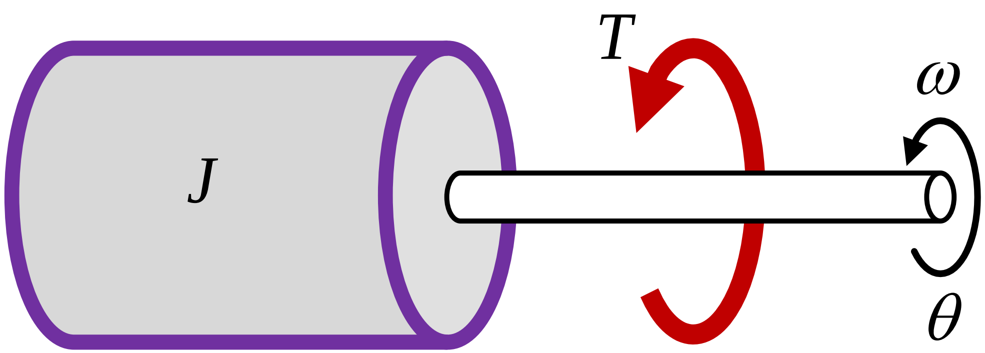
$$
\begin{cases}
F - k\left(x_2 - x_1\right) - B\left(\dot{x}_2 - \dot{x}_1\right) = m_2\,\ddot{x}_2 & (1)\\
k\left(x_2 - x_1\right) + B\left(\dot{x}_2 - \dot{x}_1\right) = m_1\,\ddot{x}_1 & (2)
\end{cases}
$$

Taking the Laplace transform yields:

$$
\begin{cases}
-\left(Bs + k\right)X_1(s) + \left(m_2 s^2 + Bs + k\right)X_2(s) = F(s) & (1)\\
\left(m_1 s^2 + Bs + k\right)X_1(s) - \left(Bs + k\right)X_2(s) = 0 & (2)
\end{cases}
$$

Mechanical Elements

- **Moment of inertia**



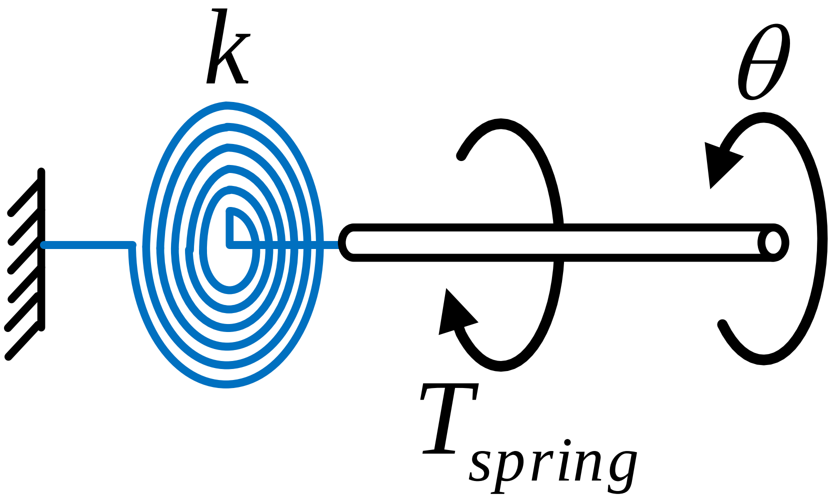
$$T = J\,\ddot{\theta}$$

$$\alpha = \dot{\omega} = \ddot{\theta}$$

where J is the moment of inertia of the body, T is the torque exerted on the body, θ is the angular position, ω is the angular velocity and α is the angular acceleration.

Mechanical Elements

- **Spiral Spring**

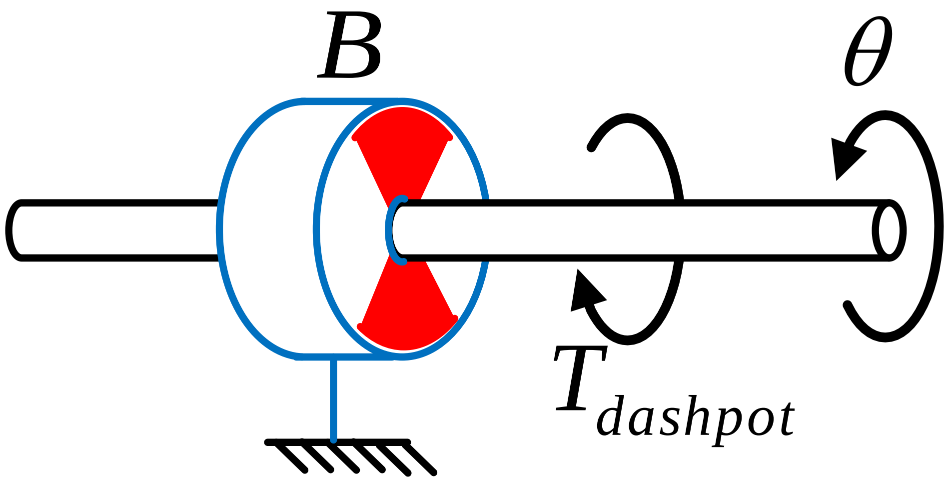


$$T_{spring} = k\,\theta$$

Where k is the spring constant.

Mechanical Elements

- **Rotational dashpot**

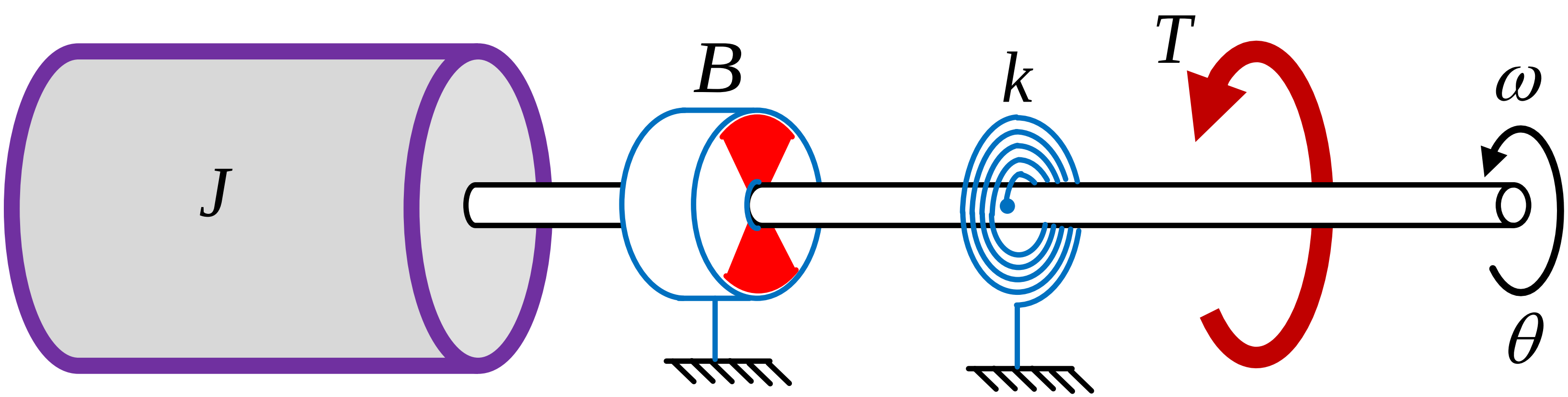


$$T_{dashpot} = B\dot{\theta}$$

Where B is the dashpot viscous friction coefficient.

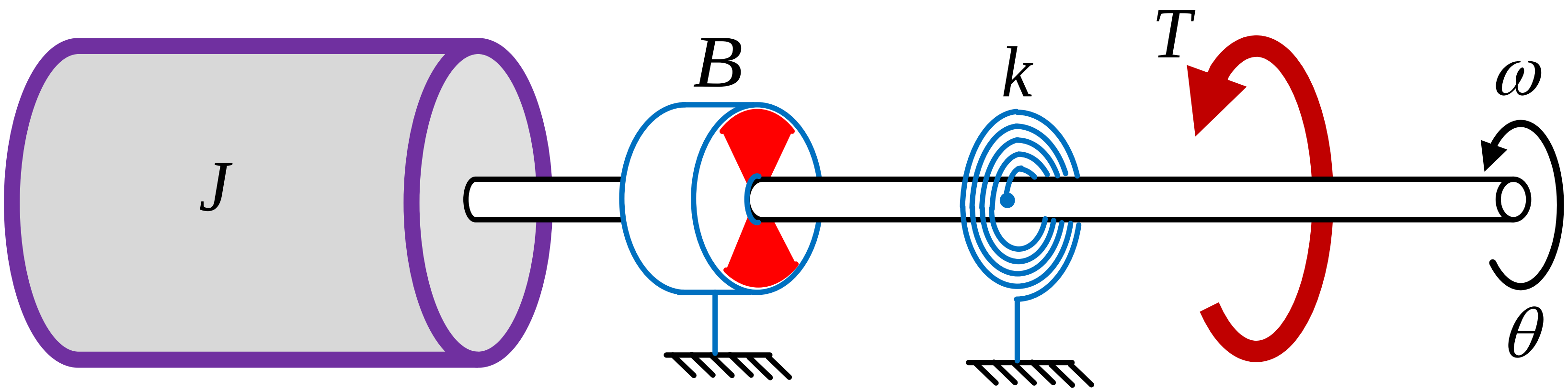
Mechanical Elements

- **Example 14:** Derive the governing equations of the following system



Mechanical Elements

- **Solution 14:**



Based on Newton's second law we have

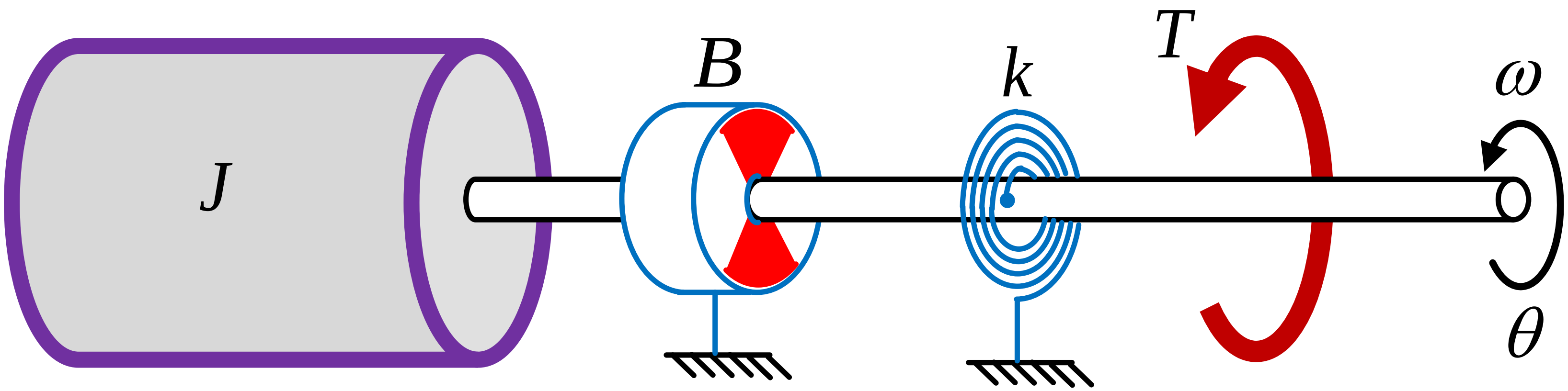
$$\sum T = J\ddot{\theta} \Rightarrow T - B\dot{\theta} - k\theta = J\ddot{\theta}$$

Taking the Laplace transform

$$T(s) = \left(Js^2 + Bs + k\right)\theta(s) \Rightarrow \frac{\theta(s)}{T(s)} = \frac{1}{\left(Js^2 + Bs + k\right)}$$

Mechanical Elements

- **Solution 14:**



If the output is rotational velocity

$$\omega = \dot{\theta} \quad \Rightarrow \quad \Omega(s) = s\Theta(s)$$

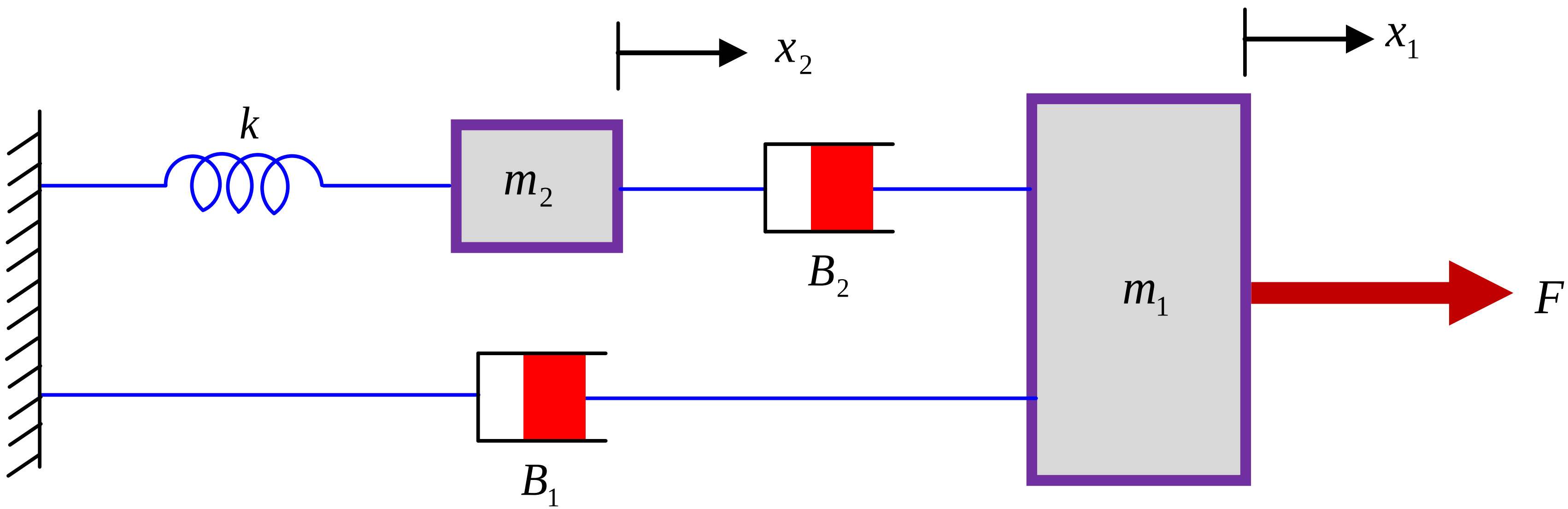
$$\Rightarrow \quad \boxed{\frac{\Omega(s)}{T(s)} = \frac{s}{\left(Js^2 + Bs + k\right)}}$$

Analogy between Mechanical and Electrical Systems

Electrical systems	Translational Mechanical Systems	Rotational Mechanical Systems
Current (i)	Force (F)	Torque (T)
Voltage (e)	Velocity (v)	Angular velocity (ω)
Magnetic flux (ϕ)	Displacement (x)	Angular position (θ)
Resistor (R)	Dashpot ($B \rightarrow 1/R$)	Dashpot ($B \rightarrow 1/R$)
Inductor (L)	Helical spring ($k \rightarrow 1/L$)	Spiral spring ($k \rightarrow 1/L$)
Capacitor (C)	Mass (m)	Moment of inertia (J)

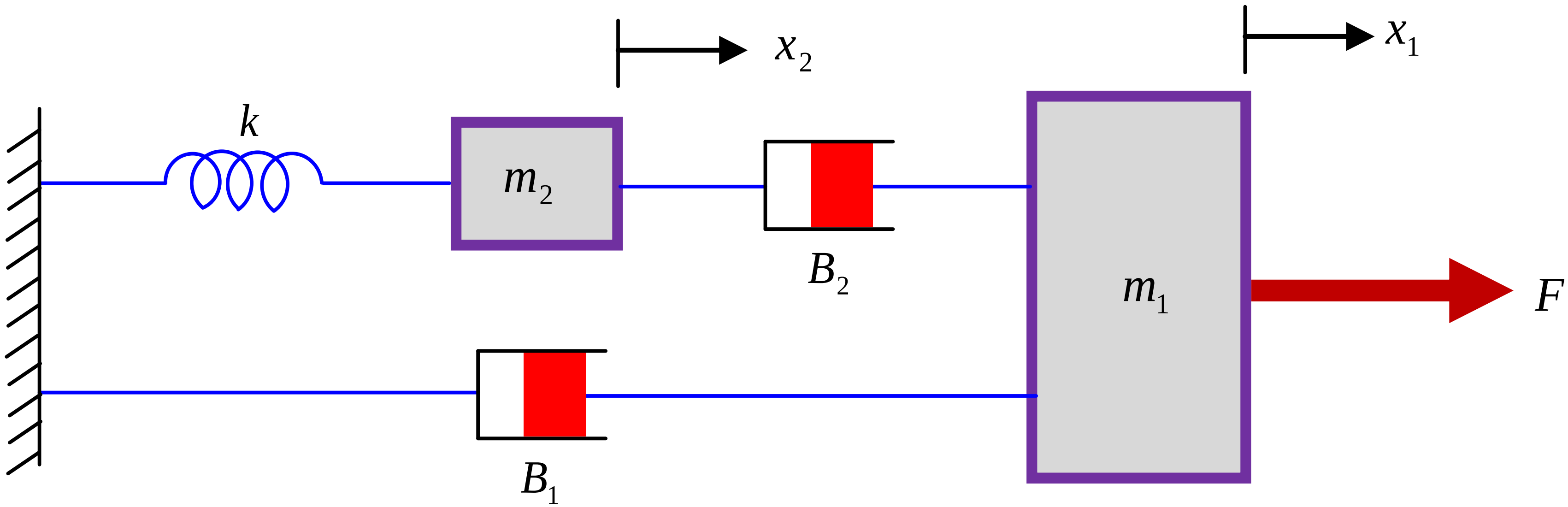
Modelling of Mechanical Systems

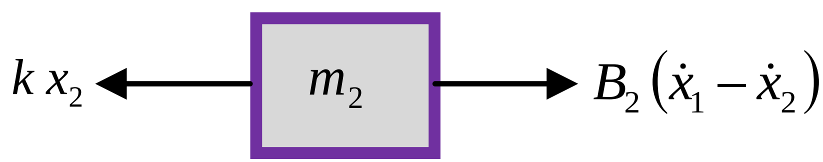
Example 15: Obtain the governing dynamic equation of the following system in which the bodies move on a frictionless surface. Also draw the electrical analogy.



Modelling of Mechanical Systems

Solution 15:

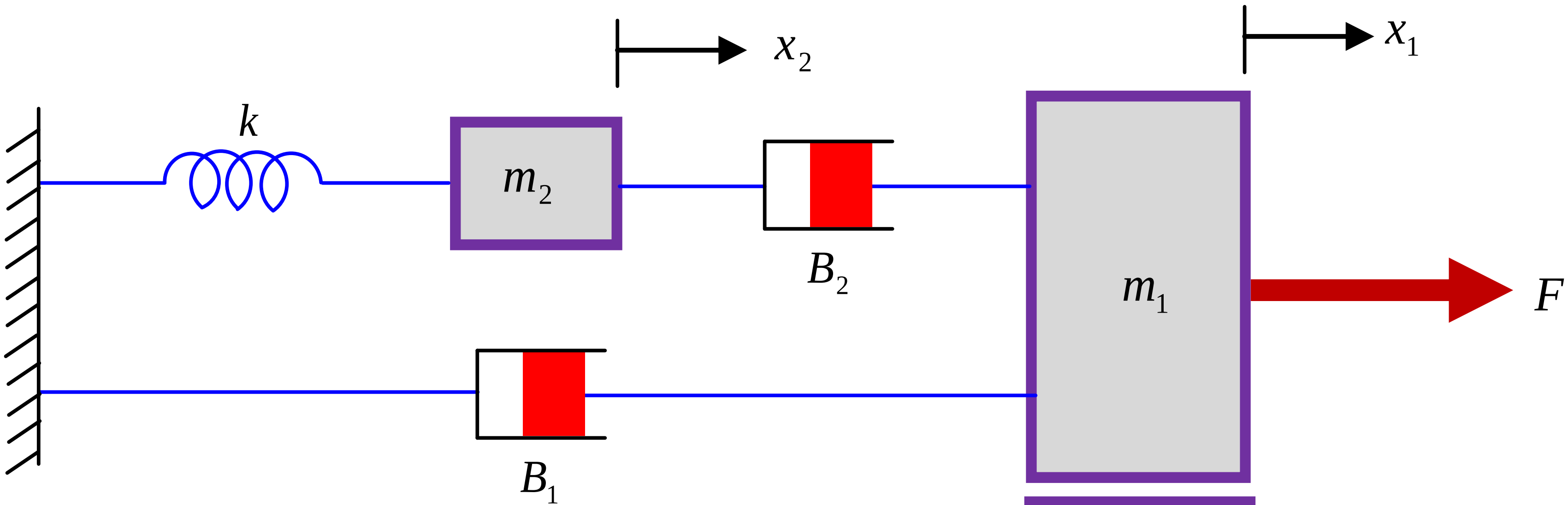


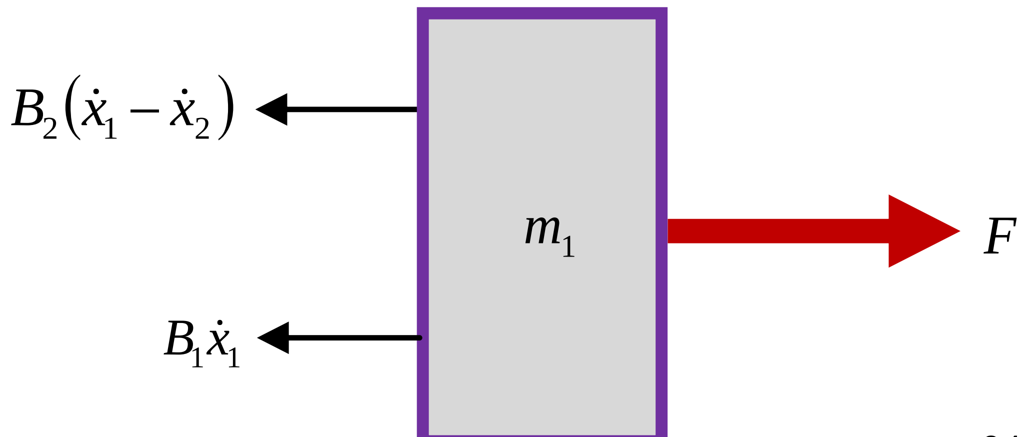


$$\sum F = m_2\ddot{x}_2 \quad \Rightarrow \quad B_2\left(\dot{x}_1-\dot{x}_2\right)-k\,x_2 = m_2\ddot{x}_2$$

Modelling of Mechanical Systems

Solution 15:



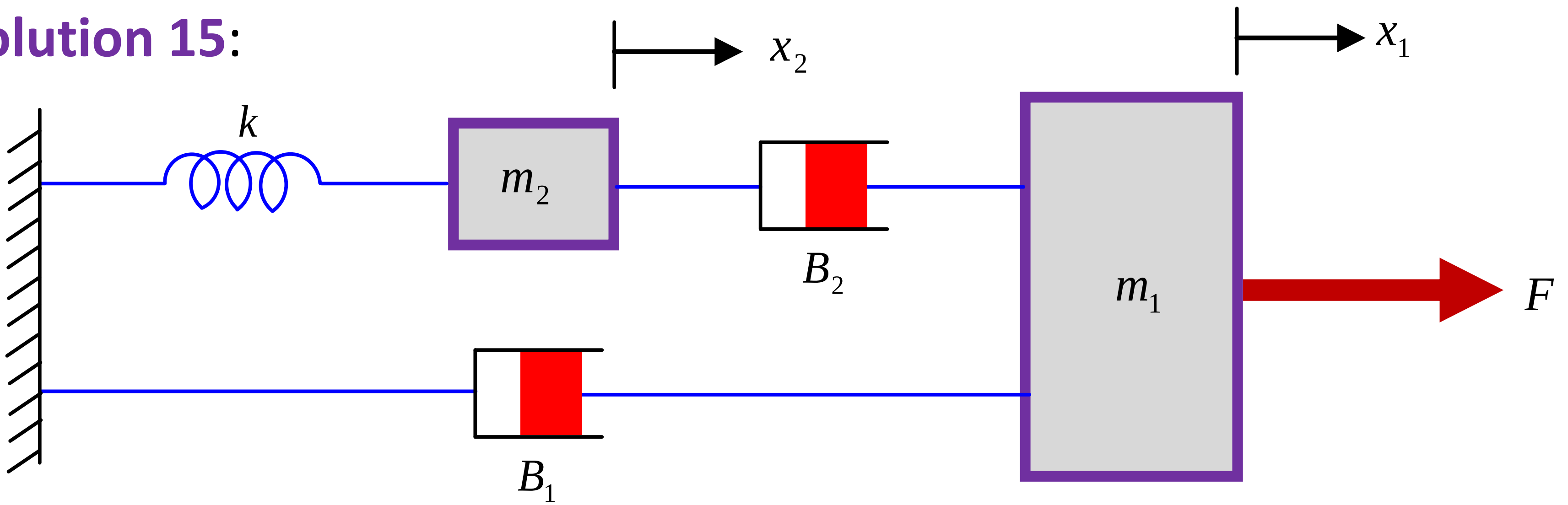


$$\sum F = m_1\ddot{x}_1$$

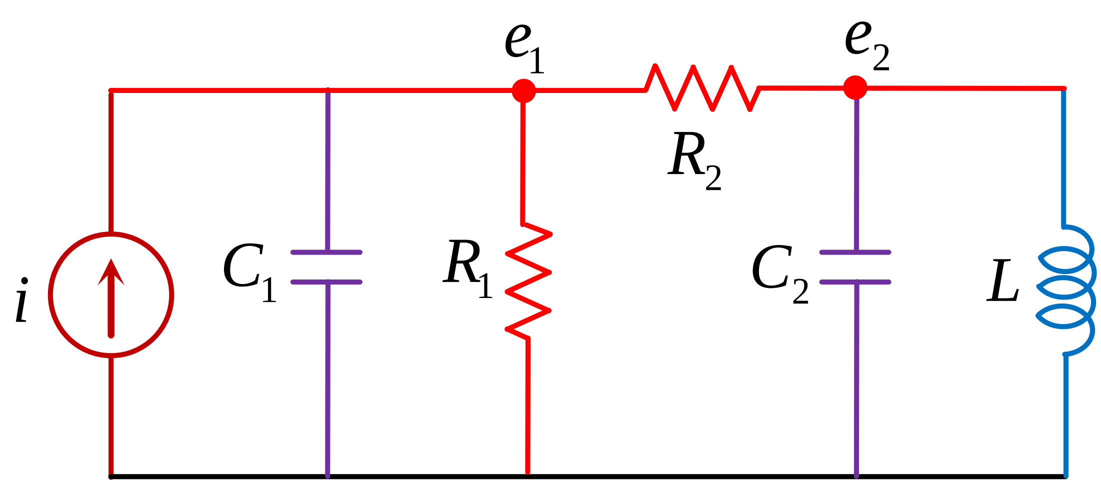
$$F - B_2(\dot{x}_1-\dot{x}_2) - B_1\dot{x}_1 = m_1\ddot{x}_1$$

Modelling of Mechanical Systems

Solution 15:



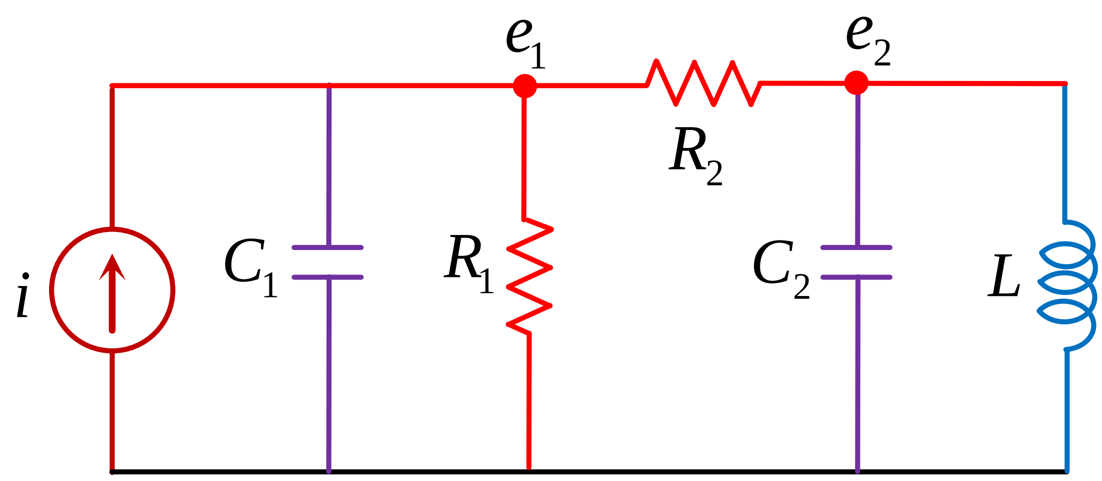
$i \equiv F$	$C_1 \equiv m_1$
$e_1 \equiv \dot{x}_1$	$C_2 \equiv m_2$
$e_2 \equiv \dot{x}_2$	$R_1 \equiv 1/B_1$
$L \equiv 1/k$	$R_2 \equiv 1/B_2$



Modelling of Mechanical Systems

Solution 15:

$i \equiv F$	$C_1 \equiv m_1$
$e_1 \equiv \dot{x}_1$	$C_2 \equiv m_2$
$e_2 \equiv \dot{x}_2$	$R_1 \equiv 1/B_1$
$L \equiv 1/k$	$R_2 \equiv 1/B_2$

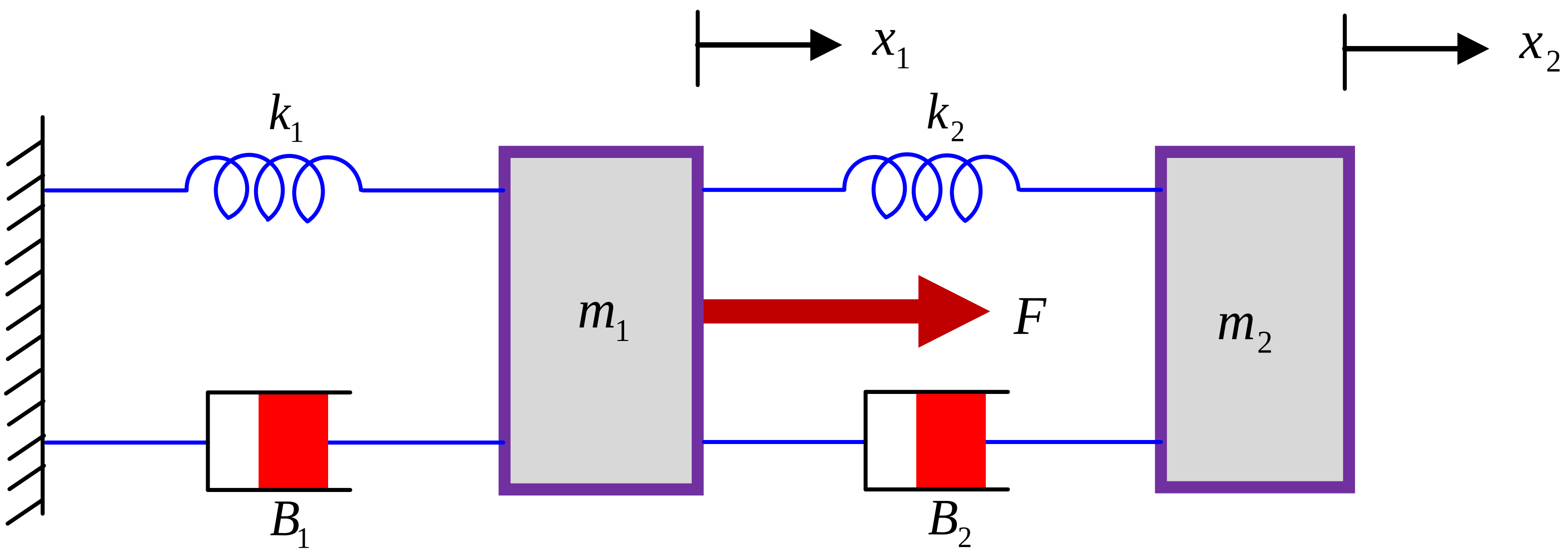


$$i - \frac{e_1 - e_2}{R_2} - \frac{e_1}{R_1} - C_1 \frac{de_1}{dt} = 0 \quad \Rightarrow \quad F - B_2\left(\dot{x}_1 - \dot{x}_2\right) - B_1 \dot{x}_1 - m_1 \ddot{x}_1 = 0$$

$$\frac{e_2 - e_1}{R_2} + C_2 \frac{de_2}{dt} + \frac{1}{L}\int_0^t e_2 dt = 0 \quad \Rightarrow \quad B_2\left(\dot{x}_1 - \dot{x}_2\right) - m_2 \ddot{x}_2 - k\, x_2 = 0$$

Modelling of Mechanical Systems

Exercise: Obtain the governing dynamic equation of the following system in which the bodies move on a frictionless surface. Also draw the electrical analogy.

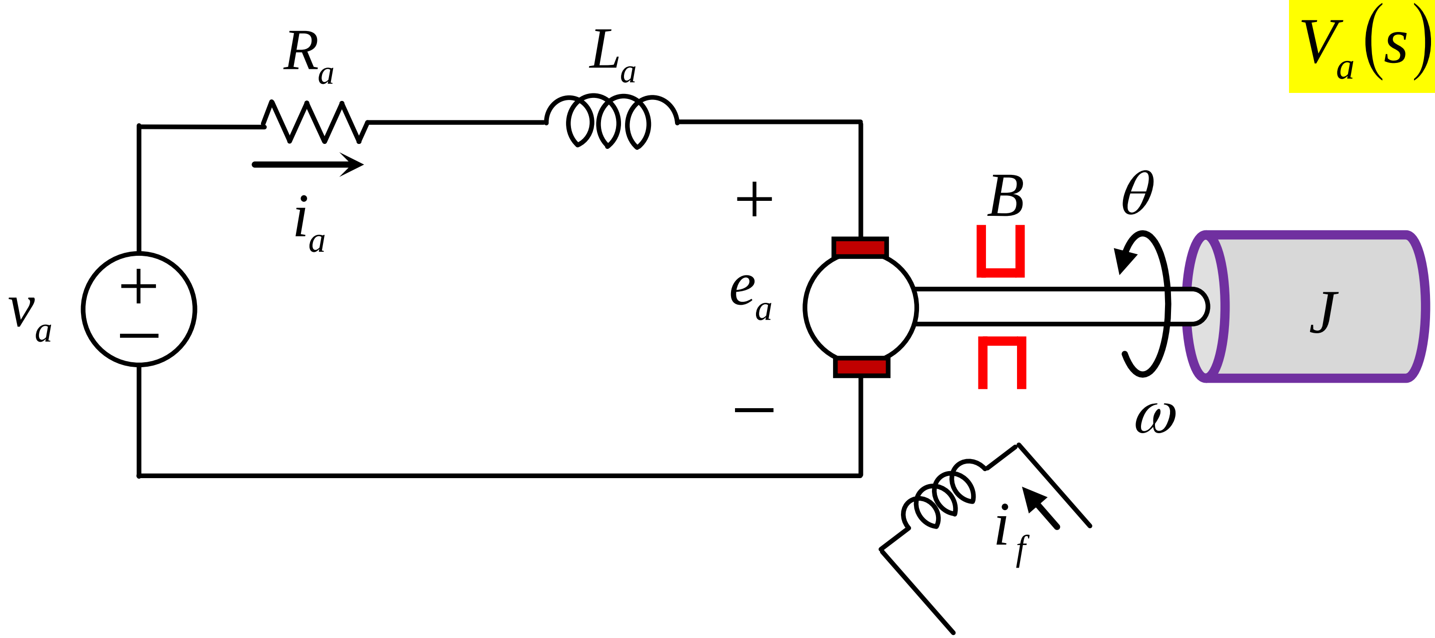


Modelling of Electromechanical Systems

DC Motors: 1-Armature Control

Assume the **armature voltage** (v_a) is the **input** to the system (DC motor) and the **angular position** (θ) of the rotor is the **output** of the system. The aim is to obtain the transfer function

$$\frac{\Theta(s)}{V_a(s)} = ?$$



Modelling of Electromechanical Systems

DC Motors: 1-Armature Control

- The relation between the developed torque and the armature current is

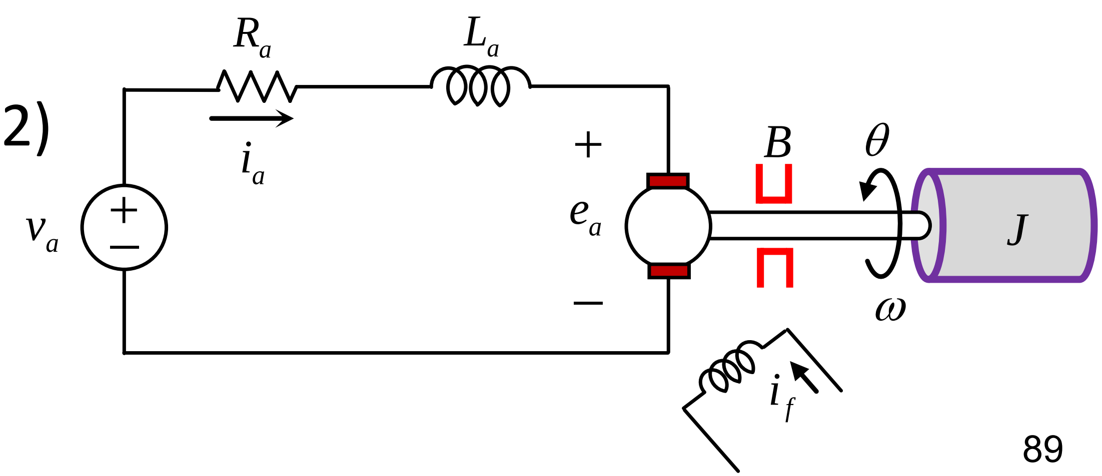
$$T = k_1 i_f i_a$$

- Since the field current is constant in the armature control technique, we have

$$T = k_t i_a \qquad (1)$$

- KVL in the armature loop:

$$v_a = R_a i_a + L_a \frac{di_a}{dt} + e_a \qquad (2)$$



Modelling of Electromechanical Systems

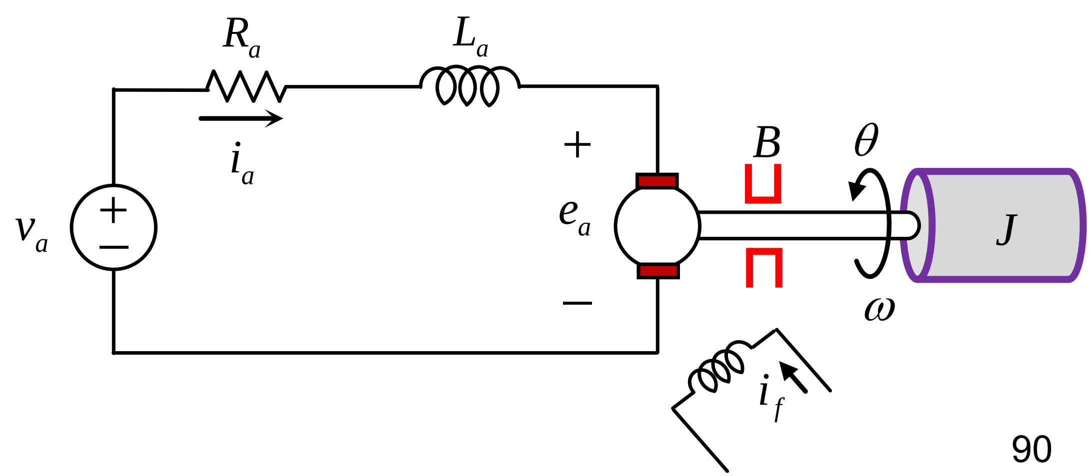
DC Motors: 1-Armature Control

- The relation between the back-emf (electromotive force) and the angular position is $e_a = k_2 i_f \omega$ $\qquad \omega = \dfrac{d\theta}{dt}$
- Since the field current is constant in the armature control technique, we have

$$e_a = k_e \dot{\theta} \qquad (3)$$

- Newton's 2nd Law

$$T = J\ddot{\theta} + B\dot{\theta} \qquad (4)$$



Modelling of Electromechanical Systems

DC Motors: 1-Armature Control

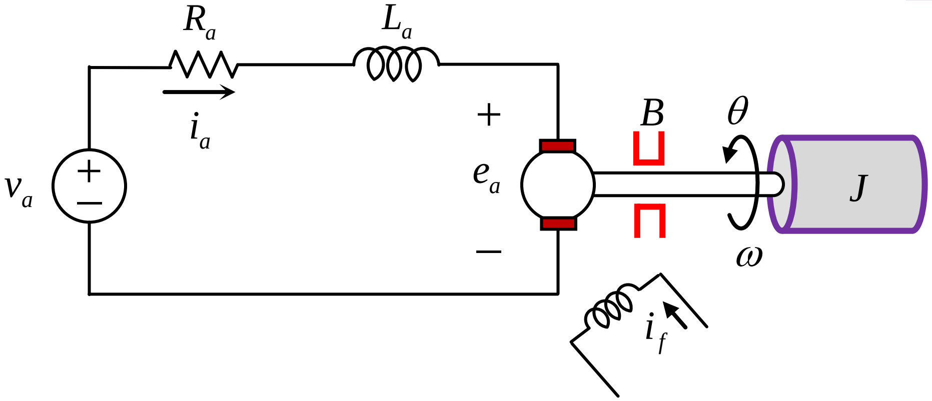
- Representing (1)-(4) and taking Laplace transform yields

$$T = k_t i_a \quad \Rightarrow \quad T(s) = k_t I_a(s) \qquad (1)$$

$$v_a = R_a i_a + L_a \frac{di_a}{dt} + e_a \quad \Rightarrow \quad V_a(s) = R_a I_a(s) + L_a s I_a(s) + E_a(s) \qquad (2)$$

$$e_a = k_e \dot{\theta} \quad \Rightarrow \quad E_a(s) = k_e s \Theta(s) \qquad (3)$$

$$T = J\ddot{\theta} + B\dot{\theta} \quad \Rightarrow \quad T(s) = Js^2\Theta(s) + Bs\Theta(s) \qquad (4)$$



Modelling of Electromechanical Systems

DC Motors: 1-Armature Control

- Representing (1)-(4) and taking Laplace transform yields

$$V_a(s) = (R_a + L_a s) I_a(s) + E_a(s)$$

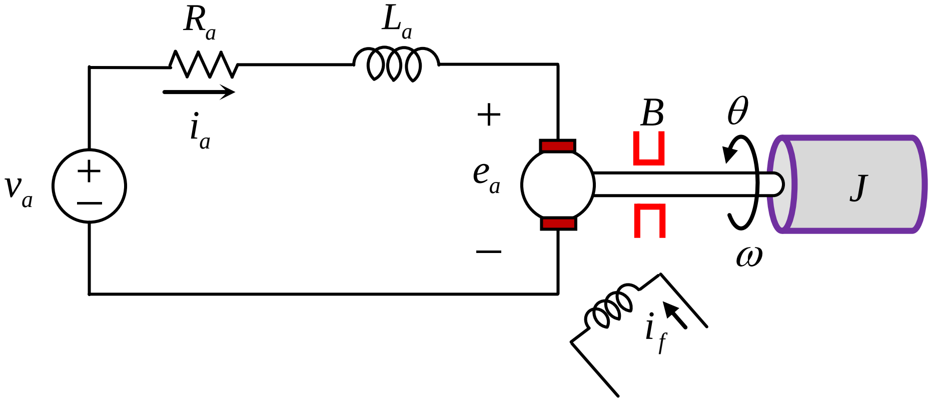
$$E_a(s) = k_e s \Theta(s)$$

$$\Rightarrow \quad V_a(s) = (R_a + L_a s) I_a(s) + k_e s \Theta(s)$$

$$T(s) = k_t I_a(s)$$

$$T(s) = (Js^2 + Bs)\Theta(s)$$

$$\Rightarrow \quad I_a(s) = \tfrac{1}{k_t}(Js^2 + Bs)\Theta(s)$$



Modelling of Electromechanical Systems

DC Motors: 1-Armature Control

$$V_a(s) = (R_a + L_a s) I_a(s) + k_e s \Theta(s)$$

$$I_a(s) = \tfrac{1}{k_t}\left(Js^2 + Bs\right)\Theta(s)$$

$$V_a(s) = \tfrac{1}{k_t}(R_a + L_a s)\left(Js^2 + Bs\right)\Theta(s) + k_e s\,\Theta(s)$$

$$\Rightarrow \quad \frac{\Theta(s)}{V_a(s)} = \frac{k_t}{(R_a + L_a s)\left(Js^2 + Bs\right) + k_t k_e s}$$

$$\Rightarrow \quad \frac{\Theta(s)}{V_a(s)} = \frac{k_t}{s\left[L_a J s^2 + (R_a J + L_a B)s + (R_a B + k_t k_e)\right]}$$

Modelling of Electromechanical Systems

DC Motors: 1-Armature Control

$$\frac{\Theta(s)}{V_a(s)} = \frac{k_t}{s\left[L_a J s^2 + (R_a J + L_a B)s + (R_a B + k_t k_e)\right]}$$

- Since L_a is normally small, the transfer function is simplified as

$$\Rightarrow \quad \frac{\Theta(s)}{V_a(s)} = \frac{k_t}{s\left[R_a J s + (R_a B + k_t k_e)\right]}$$

- If the output is the angular velocity $\Omega(s) = s\,\Theta(s)$

$$\Rightarrow \quad \frac{\Omega(s)}{V_a(s)} = \frac{k_t}{R_a J s + (R_a B + k_t k_e)} \quad \Rightarrow \quad \frac{\Omega(s)}{V_a(s)} = \frac{k_m}{\tau_m s + 1}$$

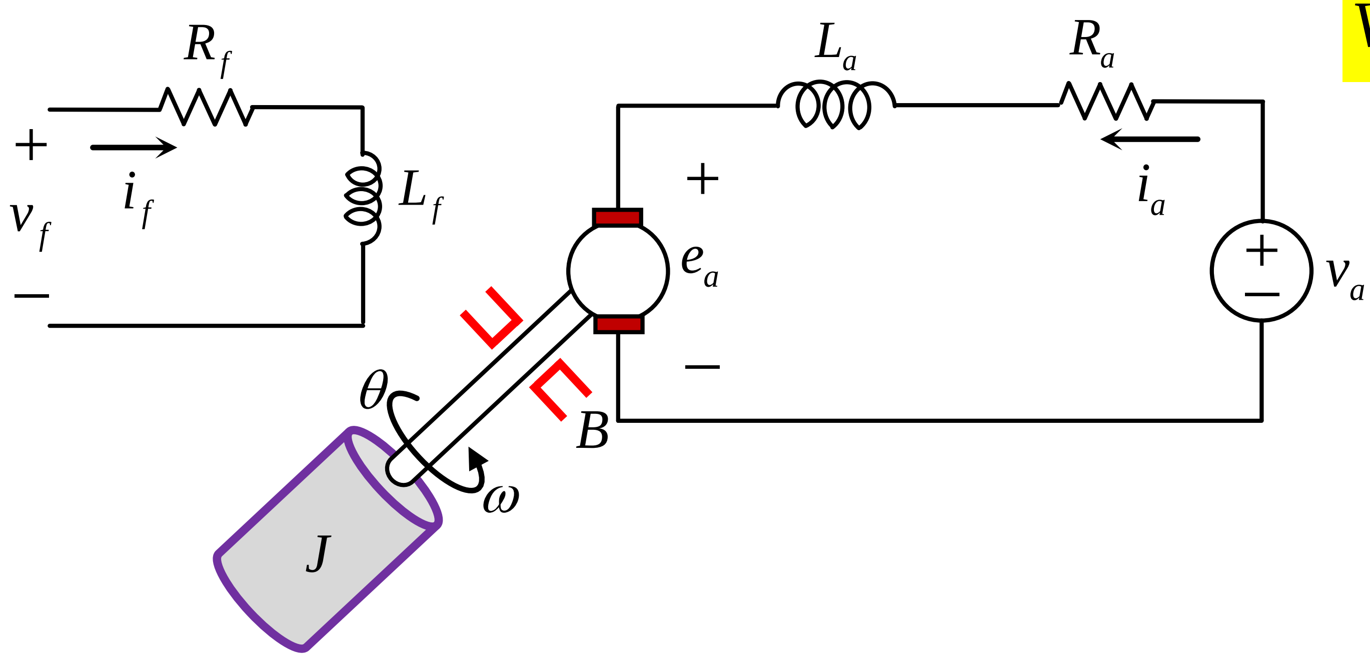
Where $k_m = \dfrac{k_t}{R_a B + k_t k_e}$ and $\tau_m = \dfrac{R_a J}{R_a B + k_t k_e}$

Modelling of Electromechanical Systems

DC Motors: 2- Field Control

Assume the **field voltage** (v_f) is the **input** to the system (DC motor) and the **angular position** (θ) of the rotor is the **output** o the system. The aim is to obtain the transfer function

$$\frac{\Theta(s)}{V_f(s)} = ?$$



Modelling of Electromechanical Systems

DC Motors: 2- Field Control

- The relation between the developed torque and the field current is

$$T = k_1 i_f i_a$$

- Since the armature current is constant in the field control technique, we have

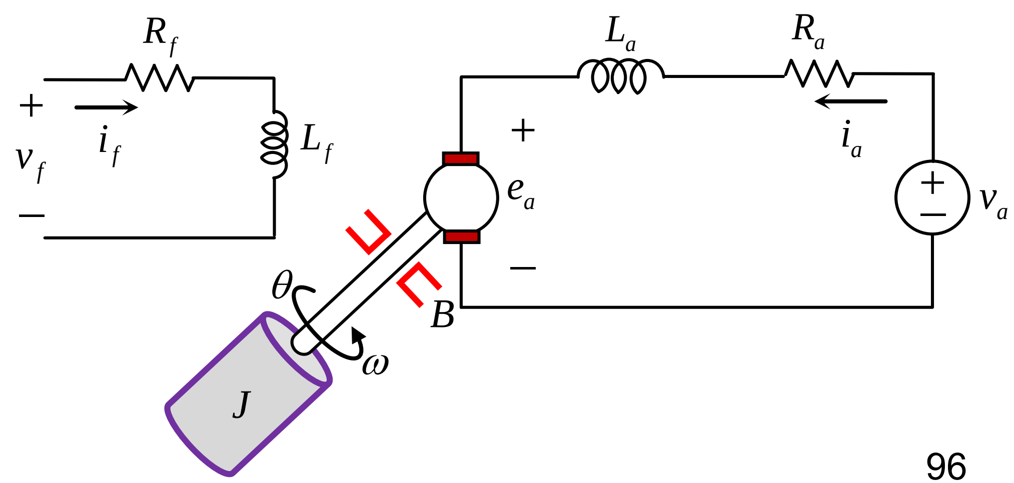
$$T = k_t i_f \qquad (1)$$

- KVL in the field loop:

$$v_f = R_f i_f + L_f \frac{di_f}{dt} \qquad (2)$$

- Newton's 2nd Law

$$T = J\ddot{\theta} + B\dot{\theta} \qquad (3)$$



Modelling of Electromechanical Systems

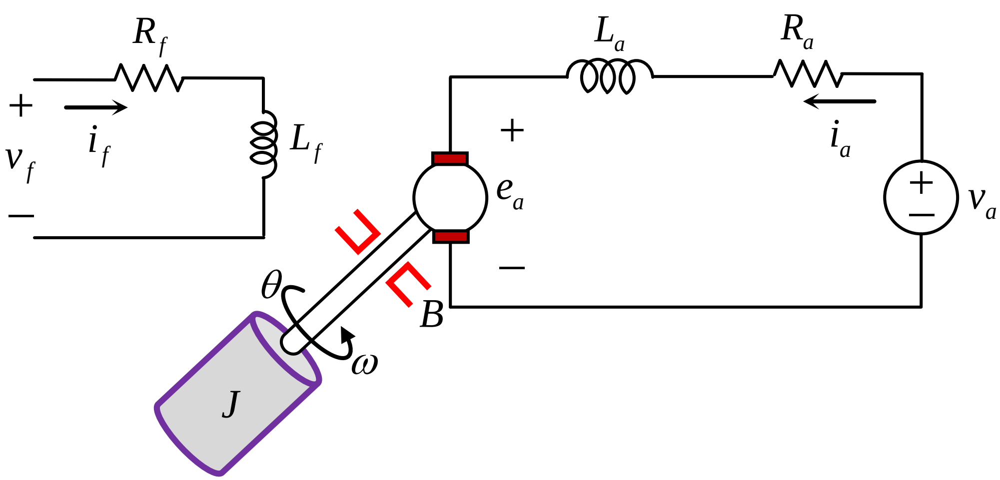
DC Motors: 2- Field Control

- Representing (1)-(3) and taking Laplace transform yields

$$T = k_t i_f \quad \Longrightarrow \quad T(s) = k_t I_f(s) \qquad (1)$$

$$v_f = R_f i_f + L_f \frac{di_f}{dt} \quad \Longrightarrow \quad V_f(s) = (R_f + L_f s) I_f(s) \qquad (2)$$

$$T = J\ddot{\theta} + B\dot{\theta} \quad \Longrightarrow \quad T(s) = (Js^2 + Bs)\Theta(s) \qquad (3)$$



$$\frac{\Theta(s)}{V_f(s)} = \frac{k_t}{s(L_f s + R_f)\,(Js + B)}$$

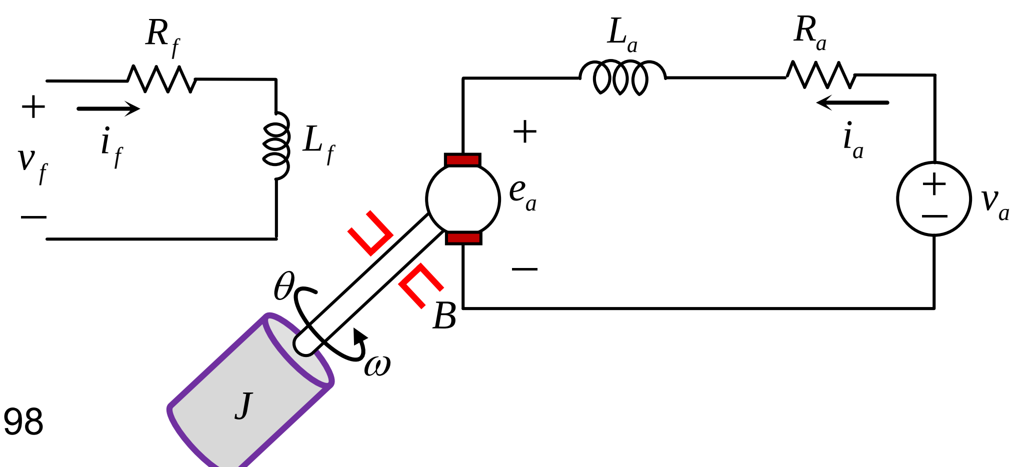
Modelling of Electromechanical Systems

DC Motors: 2- Field Control

$$\frac{\Theta(s)}{V_f(s)} = \frac{k_t}{s\left(L_f s + R_f\right)\left(Js + B\right)}$$

- If the output is angular velocity, we have $\Omega(s) = s\,\Theta(s)$

$$\Rightarrow \frac{\Omega(s)}{V_f(s)} = \frac{k_t}{\left(L_f s + R_f\right)\left(Js + B\right)} \Rightarrow \frac{\Omega(s)}{V_f(s)} = \frac{k_m}{\left(\tau_f s + 1\right)\left(\tau_m s + 1\right)}$$



where $k_m = \dfrac{k_t}{R_f B}$, $\tau_m = \dfrac{J}{B}$

and $\tau_f = \dfrac{L_f}{R_f}$

Nonlinear Systems

- A system is nonlinear if the principle of superposition does not apply.

- Thus, for a nonlinear system the response to two inputs cannot be calculated by treating one input at a time and adding the results.

- Some Examples of Nonlinear Systems:
  - The output of a component may saturate for large input signals.
  - dampers used in physical systems may be linear for low-velocity operations but may become nonlinear at high velocities

Linearization of Nonlinear Systems

- If a system operates around an **equilibrium** point and if the signals involved are **small signals**, then it is possible to approximate the nonlinear system by a linear system.
- Such a linear system is **equivalent** to the nonlinear system considered within a **limited operating range**.
- The **linearization** is based on the expansion of nonlinear function into a **Taylor series** about the operating point and the retention of only the linear term.
- Because **higher-order terms** of the Taylor series expansion are **neglected**, these neglected terms must be **small** enough. Otherwise, the result will be inaccurate.

Linearization of Nonlinear Systems

Consider a system whose input is $x(t)$ and output is $y(t)$. The relationship between $y(t)$ and $x(t)$ is given by

$$y = f(x)$$

If the normal operating condition corresponds to $x_e,\ y_e$ then above relation may be expanded into a Taylor series about this point as follows:

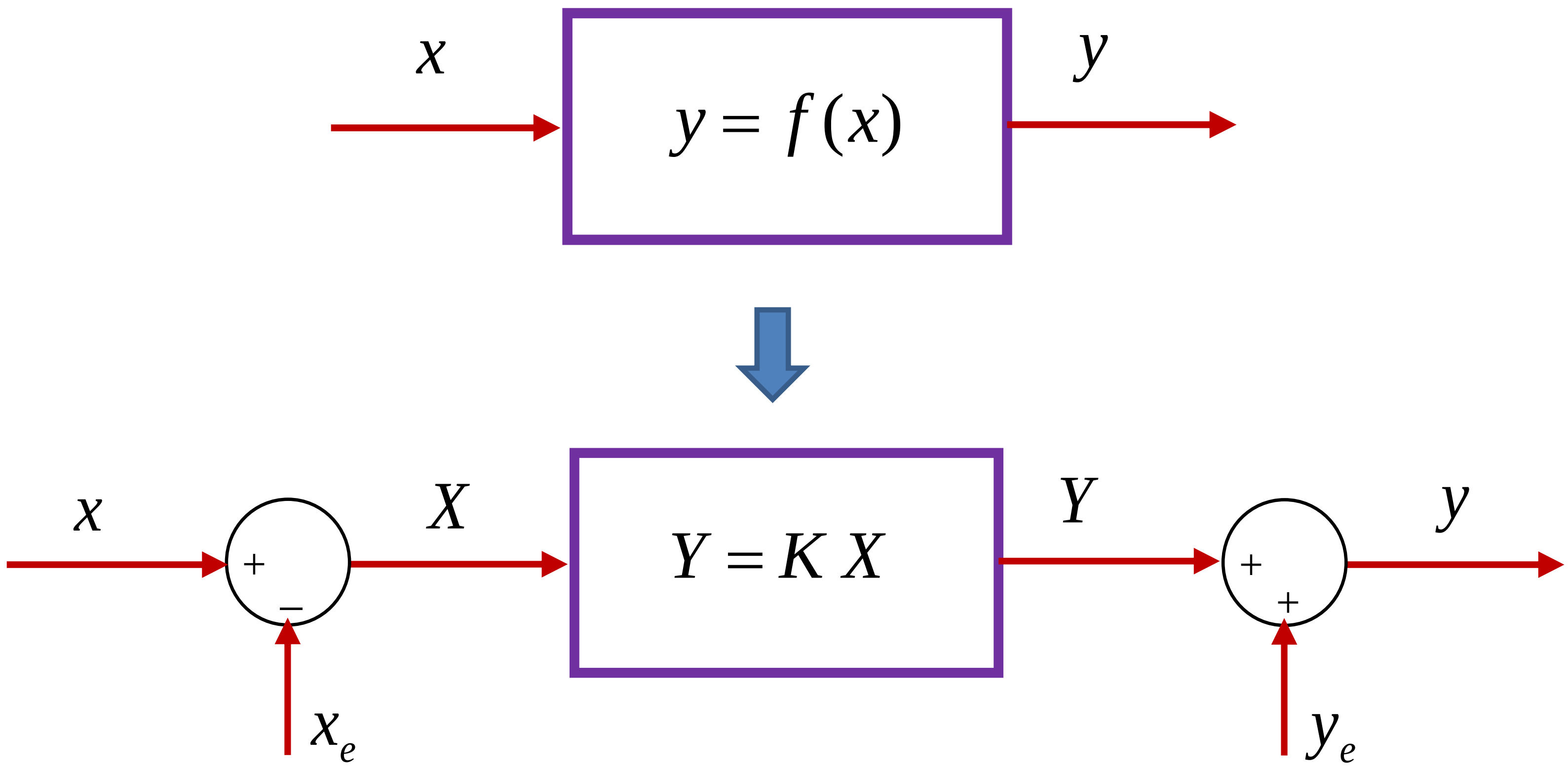
$$y = f(x_e) + \left.\frac{df}{dx}\right|_{x=x_e}(x - x_e) + \frac{1}{2!}\left.\frac{d^2 f}{dx^2}\right|_{x=x_e}(x - x_e)^2 + \cdots$$

Neglecting the higher-order terms yield $y = y_e + K(x - x_e)$

where $y_e = f(x_e)$ and $K = \left.\frac{df}{dx}\right|_{x=x_e}$

Finally assuming $X = x - x_e$ and $Y = y - y_e$ yields $Y = K\,X$

Linearization of Nonlinear Systems



$$X = x - x_e \qquad K = \left.\frac{df}{dx}\right|_{x=x_e} \qquad Y = y - y_e$$

Linearization of Nonlinear Systems

Consider a system whose input is $x_1(t)$ and $x_2(t)$ and output is $y(t)$. The relationship between $y(t)$ and the inputs is given by

$$y = f(x_1, x_2)$$

Expanding Taylor series about equilibrium point x_{1e}, x_{2e}, y_e

$$y = f(x_{1e}, x_{2e}) + \left[\left.\frac{\partial f}{\partial x_1}\right|_{\substack{x_1 = x_{1e} \\ x_2 = x_{2e}}} (x_1 - x_{1e}) + \left.\frac{\partial f}{\partial x_2}\right|_{\substack{x_1 = x_{1e} \\ x_2 = x_{2e}}} (x_2 - x_{2e}) \right]$$

$$+ \frac{1}{2!} \left[\left.\frac{\partial^2 f}{\partial x_1^2}\right|_{\substack{x_1 = x_{1e} \\ x_2 = x_{2e}}} (x_1 - x_{1e})^2 + \left.\frac{\partial^2 f}{\partial x_2^2}\right|_{\substack{x_1 = x_{1e} \\ x_2 = x_{2e}}} (x_2 - x_{2e})^2 \right.$$

$$\left. + 2 \left.\frac{\partial^2 f}{dx_1 \partial x_2}\right|_{\substack{x_1 = x_{1e} \\ x_2 = x_{2e}}} (x_1 - x_{1e})(x_2 - x_{2e}) \right] + \cdots$$

Linearization of Nonlinear Systems

Neglecting the higher-order terms yields

$$y = f(x_1, x_2) \Rightarrow y = f(x_{1e}, x_{2e}) + \left[\left.\frac{\partial f}{\partial x_1}\right|_{\substack{x_1 = x_{1e} \\ x_2 = x_{2e}}} (x_1 - x_{1e}) + \left.\frac{\partial f}{\partial x_2}\right|_{\substack{x_1 = x_{1e} \\ x_2 = x_{2e}}} (x_2 - x_{2e}) \right]$$

which can be written as $y = y_e + K_1(x_1 - x_{1e}) + K_2(x_2 - x_{2e})$

where $y_e = f(x_{1e}, x_{2e})$, $K_1 = \left.\frac{\partial f}{\partial x_1}\right|_{\substack{x_1 = x_{1e} \\ x_2 = x_{2e}}}$ and $K_2 = \left.\frac{\partial f}{\partial x_2}\right|_{\substack{x_1 = x_{1e} \\ x_2 = x_{2e}}}$

Finally assuming $X_1 = x_1 - x_{1e}$, $X_2 = x_2 - x_{2e}$ and $Y = y - y_e$ yields

$$Y = K_1 X_1 + K_2 X_2$$

Linearization of Nonlinear Systems

Linearization of the system $y = f(x_1, \ldots x_n)$ yields

$$\Rightarrow \quad y = f(x_{1e}, \ldots, x_{ne}) + \left[\left.\frac{\partial f}{\partial x_1}\right|_{\substack{x_1 = x_{1e} \\ \vdots \\ x_n = x_{ne}}} (x_1 - x_{1e}) + \cdots + \left.\frac{\partial f}{\partial x_n}\right|_{\substack{x_1 = x_{1e} \\ \vdots \\ x_n = x_{ne}}} (x_n - x_{ne}) \right]$$

which can be written as $y = y_e + K_1(x_1 - x_{1e}) + \cdots + K_n(x_n - x_{ne})$

where $y_e = f(x_{1e}, \ldots, x_{ne})$, $K_1 = \left.\frac{\partial f}{\partial x_1}\right|_{\substack{x_1 = x_{1e} \\ \vdots \\ x_n = x_{ne}}}$ and $K_n = \left.\frac{\partial f}{\partial x_n}\right|_{\substack{x_1 = x_{1e} \\ \vdots \\ x_n = x_{ne}}}$

Finally assuming $X_1 = x_1 - x_{1e}$, $X_n = x_n - x_{ne}$ and $Y = y - y_e$ yields

$$Y = K_1 X_1 + \cdots + K_n X_n$$

Linearization of Nonlinear Systems

Example 16: Linearize the following nonlinear algebraic equation about $x_{1e}=2$ and $x_{2e}=1$

$$y=2x_1^2x_2+\sin\left(\pi\, x_1\right)+\sqrt{2x_1x_2}$$

Solution:

$$\boxed{Y=K_1\,X_1+K_2\,X_2}$$

$$Y=y-y_e$$

$$X_1=x_1-x_{1e}$$

$$X_2=x_2-x_{2e}$$

$$K_1=\left.\frac{\partial f}{\partial x_1}\right|_{\substack{x_1=x_{1e}\\ x_2=x_{2e}}}$$

$$K_2=\left.\frac{\partial f}{\partial x_2}\right|_{\substack{x_1=x_{1e}\\ x_2=x_{2e}}}$$

$$y_e=f\left(x_{1e},x_{2e}\right)$$

Linearization of Nonlinear Systems

Solution 16: $x_{1e} = 2$ $\quad x_{2e} = 1$ $\quad y = 2x_1^2 x_2 + \sin(\pi x_1) + \sqrt{2x_1 x_2}$

$$Y = K_1 X_1 + K_2 X_2 \quad \Rightarrow \quad Y = (8.5 + \pi) X_1 + 9 X_2$$

$$y_e = f(x_{1e}, x_{2e}) = 2 \times 2^2 \times 1 + \sin(2\pi) + \sqrt{2 \times 2 \times 1} = 10$$

$$Y = y - y_e = y - 10 \qquad X_1 = x_1 - x_{1e} = x_1 - 2 \qquad X_2 = x_2 - x_{2e} = x_2 - 1$$

$$K_1 = \left.\frac{\partial f}{\partial x_1}\right|_{\substack{x_1 = x_{1e} \\ x_2 = x_{2e}}} = \left.\left(4x_1 x_2 + \pi \cos(\pi x_1) + \frac{x_2}{\sqrt{2x_1 x_2}}\right)\right|_{\substack{x_1 = 2 \\ x_2 = 1}} = 8 + \pi + \frac{1}{2} = (8.5 + \pi)$$

$$K_2 = \left.\frac{\partial f}{\partial x_2}\right|_{\substack{x_1 = x_{1e} \\ x_2 = x_{2e}}} = \left.\left(2x_1^2 + \frac{x_1}{\sqrt{2x_1 x_2}}\right)\right|_{\substack{x_1 = 2 \\ x_2 = 1}} = 8 + 1 = 9$$

Linearization of Nonlinear State-Space Equations

Consider a system with p inputs, q outputs and n states representing by the following nonlinear state-space equations

$$\dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}(t), \mathbf{u}(t)\right)$$
$$\mathbf{y} = \mathbf{g}\left(\mathbf{x}(t), \mathbf{u}(t)\right)$$

They can be written in the following form

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} f_1\left(x_1, \ldots, x_n, u_1, \ldots u_p\right) \\ \vdots \\ f_n\left(x_1, \ldots, x_n, u_1, \ldots u_p\right) \end{bmatrix} \qquad \begin{bmatrix} y_1 \\ \vdots \\ y_q \end{bmatrix} = \begin{bmatrix} g_1\left(x_1, \ldots, x_n, u_1, \ldots u_p\right) \\ \vdots \\ g_q\left(x_1, \ldots, x_n, u_1, \ldots u_p\right) \end{bmatrix}$$

The linearization of the system about its equilibrium $\mathbf{x} = \mathbf{x}_e$ and $\mathbf{u} = \mathbf{u}_e$ is required.

Linearization of Nonlinear State-Space Equations

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} f_1\left(x_1,\ldots,x_n,u_1,\ldots u_p\right) \\ \vdots \\ f_n\left(x_1,\ldots,x_n,u_1,\ldots u_p\right) \end{bmatrix}$$

$$\mathbf{x} = \mathbf{x}_e$$

$$\mathbf{u} = \mathbf{u}_e$$

The linearization of the nonlinear system about its equilibrium yields

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} B_{11} & \cdots & B_{1p} \\ \vdots & \ddots & \vdots \\ B_{n1} & \cdots & B_{np} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_p \end{bmatrix}$$

$$A_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{\substack{\mathbf{x}=\mathbf{x}_e \\ \mathbf{u}=\mathbf{u}_e}} \qquad B_{ik} = \left.\frac{\partial f_i}{\partial u_k}\right|_{\substack{\mathbf{x}=\mathbf{x}_e \\ \mathbf{u}=\mathbf{u}_e}}$$

$$i = 1, 2, \ldots, n$$

$$j = 1, 2, \ldots, n$$

$$k = 1, 2, \ldots, p$$

Linearization of Nonlinear State-Space Equations

$$\begin{bmatrix} y_1 \\ \vdots \\ y_q \end{bmatrix} = \begin{bmatrix} g_1\left(x_1,\ldots,x_n,u_1,\ldots u_p\right) \\ \vdots \\ g_q\left(x_1,\ldots,x_n,u_1,\ldots u_p\right) \end{bmatrix}$$

$$\mathbf{x} = \mathbf{x}_e$$

$$\mathbf{u} = \mathbf{u}_e$$

The linearization of the nonlinear system about its equilibrium yields

$$\begin{bmatrix} y_1 \\ \vdots \\ y_q \end{bmatrix} = \begin{bmatrix} C_{11} & \cdots & C_{1n} \\ \vdots & \ddots & \vdots \\ C_{q1} & \cdots & C_{qn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} D_{11} & \cdots & D_{1p} \\ \vdots & \ddots & \vdots \\ D_{q1} & \cdots & D_{qp} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_p \end{bmatrix}$$

$$C_{lj} = \left.\frac{\partial g_l}{\partial x_j}\right|_{\substack{\mathbf{x}=\mathbf{x}_e \\ \mathbf{u}=\mathbf{u}_e}} \qquad D_{lk} = \left.\frac{\partial g_l}{\partial u_k}\right|_{\substack{\mathbf{x}=\mathbf{x}_e \\ \mathbf{u}=\mathbf{u}_e}}$$

$$j = 1, 2, \ldots, n$$

$$k = 1, 2, \ldots, p$$

$$l = 1, 2, \ldots, q$$